

# Rapidity window dependences of higher order cumulants and diffusion master equation

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We study the rapidity window dependences of higher order cumulants of conserved charges observed in relativistic heavy ion collisions. The time evolution and the rapidity window dependence of the non-Gaussian fluctuations are described by the diffusion master equation. Analytic formulas for the time evolution of cumulants in a rapidity window are obtained for arbitrary initial conditions. We discuss that the rapidity window dependences of the non-Gaussian cumulants have characteristic structures reflecting the non-equilibrium property of fluctuations, which can be observed in relativistic heavy ion collisions with the present detectors. It is argued that various information on the thermal and transport properties of the hot medium can be revealed experimentally by the study of the rapidity window dependences, especially by the combined use, of the higher order cumulants. Formulas of higher order cumulants for a probability distribution composed of sub-probabilities, which are useful for various studies of non-Gaussian cumulants, are also presented.

## I. INTRODUCTION

Bulk fluctuations of conserved charges are believed to be useful observables for the study of thermal property of the primordial medium created in relativistic heavy ion collisions. Active experimental analyses of fluctuation observables have been performed in the beam-energy scan (BES) program [1] at the Relativistic Heavy Ion Collider (RHIC) [2–4], as well as the Large Hadron Collider (LHC) [5]. These experiments have been observed not only the second order but also higher order cumulants characterizing non-Gaussianity of the fluctuations by the event-by-event analysis. Various theoretical suggestions have been made to utilize fluctuation observables as signals of the deconfinement transition and the QCD critical point [6–21]. The cumulants of conserved charges in the equilibrated medium are also observable in lattice QCD Monte Carlo simulations, and the numerical study of various cumulants have been carried out actively [22–30].

One of the most notable features of the measurement of fluctuations in heavy ion collisions is that higher order cumulants of particle numbers in a rapidity window have been observed with moderate statistics up to fourth order. Such measurements are possible because the systems observed in these experiments are not large; the particle number measured in one event is at most of order  $10^3$  [2, 5], although the local equilibration is expected to realize for some period in the time evolution. Experimental detectors are designed to observe and identify almost all charged particles entering the detector. They thus can provide *full counting statistics* [31] of the event-by-event distribution of the particle numbers up to the error arising from the acceptance, efficiency and particle miss identifications [17, 32–35] of the detectors.

Because the experimental detectors in heavy ion collisions count particle numbers which arrive at the detectors, the event-by-event analysis can observe fluctuations in the final state of the collision events. If the fluctuations are fully equilibrated in the final state, they can only tell us information on the thermodynamics at this time, which is in the hadronic medium and well described by the hadron resonance gas (HRG) model [13]. The experimental results, however, suggest that the observed fluctuations are not the one of the equilibrated hadronic medium. First, the experimental results on the cumulants and their ratios have statistically-significant deviation from the values in the HRG model [2, 5]. In the HRG model, the fluctuations of net particle numbers, including net-baryon and net-electric charge which are conserved charges, are given by the Skellam distribution to a good approximation [13]. The cumulants observed at RHIC and LHC show deviations from these Skellam values. The second observation is concerned with the rapidity window dependences of the cumulants. In a thermal medium in grand canonical ensemble, cumulants of conserved charges are extensive variables and proportional to volume. In heavy ion collisions, fluctuations of particle number in a given pseudo-rapidity window,  $\Delta\eta$ , are observed. Assuming the Bjorken expansion and that the pseudo-rapidity window can be used as a proxy of the one of the coordinate-space rapidity,  $\Delta y$  [36], the spatial volume to count the particle number is proportional to  $\Delta\eta$ . The cumulants, therefore, should be proportional to  $\Delta\eta$  if the fluctuations are fully equilibrated. The net-electric charge fluctuation  $\langle(\delta N_Q)^2\rangle$  observed by ALICE collaboration [5], however, clearly shows that this proportionality is violated. A similar violation is also indicated in the higher order cumulants of net-proton number observed by STAR collaboration [4]. It is this non-equilibrium nature of fluctuations which enables us to explore the primordial thermodynamics using fluctuation observables [8, 9].

For conserved charges, the off-equilibrium property of fluctuations in the final state is not surprising, because the modification of local density of a conserved charge is proceeded only by the diffusive processes, which is typically slow; because of the conservation law, the time scale of the diffusive process can become arbitrary slow as the spatial scale becomes larger. In fact, earlier studies suggested that the time scale of the diffusive process for conserved charges is so slow that the thermal fluctuations generated in the deconfined medium survive until the final state when  $\Delta\eta$  is taken sufficiently large [8, 9, 37, 38]. Because the effect of the diffusion with a fixed  $\Delta\eta$  is determined by the transport of the charges, the conservation law also suggests that one can investigate the transport property of the hot medium, such as the magnitude of the diffusion constant, experimentally using the  $\Delta\eta$  dependence of conserved-charge fluctuations [5, 19, 37].

In Refs. [8, 9, 37, 38], the time evolution of fluctuations are considered only for Gaussian fluctuations. Because the heavy ion collisions can measure the higher order cumulants, the  $\Delta\eta$  dependences of not only Gaussian but also non-Gaussian fluctuations should encode more information on the thermal and transport properties of the medium [19]. To extract such information, however, an appropriate description of the non-equilibrium time evolution of the non-Gaussian fluctuations is required.

When one investigates the diffusive process of non-Gaussian fluctuations in heavy ion collisions, it is important to keep in mind that the cumulants observed so far are not far from the ones of the Skellam distribution [2, 5], which are the equilibrated values in the HRG model. A possible interpretation of these results is that the signals in cumulants are developed in the primordial stage, but they are blurred by the diffusive process in the hadronic stage. In fact, the experimental result in Ref. [5] indicates that the fluctuations *increase* toward the equilibrated value in the hadronic medium due to the diffusion [19, 21, 37]. If this is the case, it is important to describe the approach of the cumulants toward the hadronic value in the analysis of the diffusive process in the hadronic medium. For this purpose, the model

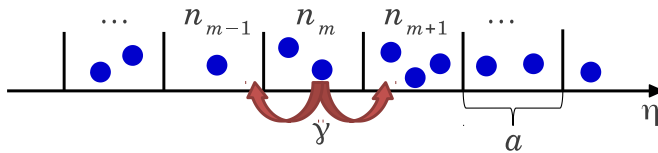


FIG. 1. System described by the diffusion master equation (DME) Eq. (1).

for the diffusive process should have nonzero higher order cumulants in equilibrium which are consistent with those in the hadronic medium.

In Ref. [19], to describe the time evolution of non-Gaussian fluctuations the diffusion master equation (DME) is employed. As discussed in Ref. [19], in this model the fluctuations in equilibrium are given by the Poisson or Skellam distribution as a consequence of the discrete nature of the particle number. The time evolutions of the second and fourth order cumulants of net particle numbers are studied in this model. It is found that the fourth-order cumulant can show characteristic behaviors as a function of  $\Delta\eta$  and tends to be suppressed compared with the second order one, if the thermal fluctuations in the primordial medium are suppressed. These results can be confirmed experimentally with the present detectors. The analysis is later extended to finite volume systems [21] to investigate the effect of the global charge conservation.

The purpose of the present paper is to elaborate on detailed discussions skipped in Ref. [19]. In the present study, we show the analytic procedure for obtaining the time evolution of cumulants in the DME with arbitrary initial conditions. We also extend the discussion on the  $\Delta\eta$  dependence of cumulants in Ref. [19] to more general cases. The  $\Delta\eta$  dependence of the third order cumulant is considered in addition to second and fourth order ones discussed in Ref. [19]. The initial condition is also extended to more general cases. Using these results, we pursue the possibility to extract various information on the thermal and transport properties of the hot medium from the non-Gaussian cumulants of net-baryon number and net-electric charge in heavy ion collisions.

This paper is organized as follows. In Sec. II, we present the analytic solution for the time evolution of cumulants in the DME. The result is then extended in Sec. III to the multi-particle systems in order to describe the cumulants of net particle numbers. In Sec. IV, we apply these results to the description of  $\Delta\eta$  dependences of non-Gaussian cumulants in heavy ion collisions. Some discussions and a short summary is given in Sec. V. In Appendix A, we derive formulas of cumulants for a probability given by a superposition of sub-probabilities, which are used in Sec. II to obtain the solution of the DME for arbitrary initial conditions. In Appendix B, we consider a simple stochastic model for a chemical reaction. This appendix will help readers who are not familiar with stochastic models like the DME to understand Sec. II.

## II. DIFFUSION MASTER EQUATION

To describe the diffusion of non-Gaussian fluctuations of the conserved charges in heavy ion collisions, we use the diffusion master equation (DME) [19]. After introducing the DME and describing its general property in Sec. II A, we derive the analytic form of the time evolution of the cumulants for fixed initial conditions in Secs. II B - II F. The solution is then extended to general initial conditions in Sec. II G.

### A. Diffusion master equation

We consider a single species of classical Brownian particles moving in one dimensional space. To describe microscopic states of this system, we divide the spatial coordinate into discrete cells with an equal length  $a$ , as in Fig. 1. We label each cell by an integer  $m$  and denote the number of particles in each cell by  $n_m$ . A microscopic state is then specified by the vector  $\mathbf{n} = (\dots, n_{m-1}, n_m, n_{m+1}, \dots)$  representing particle numbers in all cells. One can also introduce the probability distribution function  $P(\mathbf{n}, t)$  that each cell contains  $n_m$  particles at time  $t$ .

Next, we assume that each particle moves adjacent cells with a probability  $\gamma(t)$  per unit time. The probability

$P(\mathbf{n}, t)$  then follows the first-order differential equation [39]

$$\frac{\partial}{\partial t} P(\mathbf{n}, t) = \gamma(t) \sum_m [(n_m + 1) \{P(\mathbf{n} + \mathbf{e}_{m+1} - \mathbf{e}_{m-1}) + P(\mathbf{n} - \mathbf{e}_{m+1} + \mathbf{e}_{m-1})\} - 2n_m P(\mathbf{n}, t)], \quad (1)$$

where  $\mathbf{e}_m = (0, \dots, 0, 1, 0, \dots, 0)$  is the vector that all components are zero except for the  $m$ th-one which takes unity. After solving Eq. (1), we take the continuum limit defined by  $a \rightarrow 0$  to recover the continuity of the spatial coordinate. Although the generalization of the model to a multi-dimensional system is straightforward, in this study we limit our attention to the one-dimensional problem to avoid unnecessary complexity. Note that the one-dimensional model is sufficient for the description of the diffusive process along the rapidity direction in heavy ion collisions.

As we will see later, after taking the continuum limit the time evolution of the average particle number density  $n(x, t)$  (the exact definition will be given in Sec. II D) in the DME obeys the standard diffusion equation

$$\frac{\partial}{\partial t} n(x, t) = D(t) \frac{\partial^2}{\partial x^2} n(x, t), \quad (2)$$

with the diffusion constant  $D(t) = \gamma(t)a^2$ . It is also shown [19] that the time evolution of the Gaussian fluctuations of  $n(x, t)$  in this limit is consistent with the *stochastic diffusion equation*

$$\frac{\partial}{\partial t} n(x, t) = \frac{\partial}{\partial x} \left( D(t) \frac{\partial}{\partial x} n(x, t) + \xi(x, t) \right), \quad (3)$$

where  $\xi(x, t)$  represents the stochastic term. Here, it is assumed that  $\xi(x, t)$  is local with respect to  $x$  and  $t$ , i.e.  $\langle \xi(x_1, t_1) \xi(x_2, t_2) \rangle$  is proportional to  $\delta(x_1 - x_2) \delta(t_1 - t_2)$ . In this case the property of  $\xi(x, t)$  is completely determined by the fluctuation dissipation relation [40].

The DME, on the other hand, has a property that the higher order cumulants take nonzero values in equilibrium defined by  $t \rightarrow \infty$  limit. This property is contrasted with Eq. (3) in which the fluctuation of  $n(x, t)$  becomes Gaussian and all the cumulants of  $n(x, t)$  higher than the second order vanish in equilibrium for  $n$  independent  $D(t)$  [19]. The higher order cumulants of the particle number in some range of the spatial coordinate in the DME in equilibrium are given by the Poissonian ones [19]. This property is easily understood from the fact that in equilibrium each Brownian particle exists any place with an equal probability and the individual particles are uncorrelated. When the model is extended to treat the net-particle number in Sec. III, the distribution is given by the Skellam one. This property is suitable to model the time evolution of higher order cumulants in heavy ion collisions.

## B. Factorial generating function for fixed initial condition

In the following, we solve the DME Eq. (1). For readers who are not familiar with stochastic equations like Eq. (1), we recommend them to read Appendix B before this section.

In the following five subsections, we first solve Eq. (1) for the fixed initial condition,

$$P(\mathbf{n}, 0) = \prod_m \delta_{n_m, M_m}, \quad (4)$$

and present  $t$  dependence of cumulants of  $P(\mathbf{n}, t)$ . In Eq. (4), the particle number in each cell,  $n_m$ , are fixed to  $M_m$  without fluctuations at  $t = 0$ . The analysis is later generalized to initial conditions having fluctuations in Sec. II G.

To solve Eq. (1) with the initial condition Eq. (4), it is convenient to use the factorial generating function [39]

$$G_f(\mathbf{s}, t) = \sum_{\mathbf{n}} \left( \prod_m s_m^{n_m} \right) P(\mathbf{n}, t), \quad (5)$$

where the sum is taken over all possible combinations of  $n_m$ . Substituting Eq. (5) in Eq. (1), one obtains

$$\frac{\partial}{\partial t} G_f(\mathbf{s}, t) = \gamma(t) \sum_m (s_{m+1} - 2s_m + s_{m-1}) \frac{\partial}{\partial s_m} G_f(\mathbf{s}, t). \quad (6)$$

Because Eq. (6) is a first-order partial differential equation, this equation is solved by the method of characteristics. In this method, we use the fact that the solution of Eq. (6) satisfies

$$\frac{d}{dt} G_f(\mathbf{s}^c(t), t) = 0, \quad (7)$$

where the characteristic line  $s_m^c(t)$  is the solution of the characteristic equations

$$\frac{ds_m^c}{dt} = -\gamma(t)(s_{m+1}^c - 2s_m^c + s_{m-1}^c). \quad (8)$$

Equation (8) is easily solved in Fourier space. Assuming the periodic boundary condition and defining

$$r_j = \frac{1}{N} \sum_m s_m e^{2\pi i j m / N}, \quad (9)$$

with  $N$  being the total number of cells, the characteristic equations become

$$\frac{d}{dt} r_j^c = -\gamma(t)(e^{2\pi i j / N} + e^{-2\pi i j / N} - 2)r_j^c \equiv \omega_j(t)r_j^c, \quad (10)$$

with

$$\omega_j(t) = -\gamma(t)(e^{2\pi i j / N} + e^{-2\pi i j / N} - 2) \simeq \gamma(t) \left( \frac{2\pi}{N} j \right)^2, \quad (11)$$

where the last nearly-equality is satisfied for  $j/N \ll 1$ . The solution of Eq. (10) is

$$r_j^c(t) = r_j^c(0) \exp\left[\int_0^t dt' \omega_j(t')\right]. \quad (12)$$

When  $\gamma(t)$  does not have  $t$  dependence, Eq. (12) has a simple form

$$r_j^c(t) = r_j^c(0) e^{\omega_j t}. \quad (13)$$

The initial condition Eq. (4) is transferred to the factorial generating function  $G_f(\mathbf{s}, t)$  as

$$G_f(\mathbf{s}, 0) = \prod_m s_m^{M_m} = \prod_m \left( \sum_j r_j(0) e^{2\pi i j m / N} \right)^{M_m}. \quad (14)$$

Because of Eq. (7), the solution of Eq. (6) is obtained by substituting Eq. (12) in Eq. (14) which yields

$$G_f(\mathbf{s}, t) = \prod_m \left( \sum_j r_j e^{2\pi i j m / N} \exp\left[-\int_0^t dt' \omega_j(t')\right] \right)^{M_m}. \quad (15)$$

The corresponding factorial cumulant generating function is given by

$$K_f(\mathbf{s}, t) = \log G_f(\mathbf{s}, t) = \sum_m M_m \log \sum_j r_j e^{2\pi i j m / N} \exp\left[-\int_0^t dt' \omega_j(t')\right]. \quad (16)$$

### C. Cumulants and factorial cumulants

Using Eq. (16), the factorial cumulants of  $n_m$  are given by

$$\langle n_{m_1} n_{m_2} \cdots n_{m_l} \rangle_{fc} = \left. \frac{\partial^l K_f}{\partial s_{m_1} \partial s_{m_2} \cdots \partial s_{m_l}} \right|_{\mathbf{s}=\mathbf{1}}, \quad (17)$$

where  $\mathbf{s} = \mathbf{1}$  means that  $s_m = 1$  for all  $m$ . Note that this condition corresponds to  $r_0 = 1$  and  $r_j = 0$  for  $j \neq 0$ . Using this relation, factorial cumulants in the Fourier space are calculated to be

$$\begin{aligned} \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \cdots \tilde{n}_{k_l} \rangle_{fc} &= \sum_{m_1, \dots, m_l} e^{-2\pi i (k_1 m_1 + \cdots + k_l m_l) / N} \langle \tilde{n}_{m_1} \cdots \tilde{n}_{m_l} \rangle_{fc} \\ &= \sum_{m_1, \dots, m_l} e^{-2\pi i (k_1 m_1 + \cdots + k_l m_l) / N} \left. \frac{\partial^l K_f}{\partial s_{m_1} \cdots \partial s_{m_l}} \right|_{\mathbf{s}=\mathbf{1}} \\ &= \left. \frac{\partial^l K_f}{\partial r_{k_1} \cdots \partial r_{k_l}} \right|_{\mathbf{s}=\mathbf{1}}, \end{aligned} \quad (18)$$

where the Fourier transform of the particle number is defined by

$$\tilde{n}_k = \sum_x n_m e^{-2\pi i k m / N}, \quad n_m = \frac{1}{N} \sum_k \tilde{n}_k e^{2\pi i k m / N}. \quad (19)$$

Substituting Eq. (16) in Eq. (18), one obtains the factorial cumulants up to fourth order as

$$\langle \tilde{n}_k \rangle_{\text{fc}} = \tilde{M}_k \exp[-\int_0^t dt' \omega_k(t')], \quad (20)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \rangle_{\text{fc}} = -\tilde{M}_{k_1+k_2} \exp[-\int_0^t dt' (\omega_{k_1}(t') + \omega_{k_2}(t'))], \quad (21)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \rangle_{\text{fc}} = 2\tilde{M}_{k_1+k_2+k_3} \exp[-\int_0^t dt' (\omega_{k_1}(t') + \omega_{k_2}(t') + \omega_{k_3}(t'))], \quad (22)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \tilde{n}_{k_4} \rangle_{\text{fc}} = -6\tilde{M}_{k_1+k_2+k_3+k_4} \exp[-\int_0^t dt' (\omega_{k_1}(t') + \omega_{k_2}(t') + \omega_{k_3}(t') + \omega_{k_4}(t'))], \quad (23)$$

with  $\tilde{M}_k = \sum_m M_m e^{-2\pi i k m / N}$ . Although we do not show the explicit forms of cumulants higher than fourth order, they can be obtained straightforwardly with a similar manipulation.

Cumulants of  $\tilde{n}_k$  are obtained by using cumulant generating function

$$K(\boldsymbol{\theta}, t) = K_{\text{f}}(\mathbf{s}, t)|_{\mathbf{s}_i = e^{\theta_i}}, \quad (24)$$

as

$$\langle n_{m_1} n_{m_2} \cdots n_{m_l} \rangle_{\text{c}} = \frac{\partial^l K}{\partial \theta_1 \cdots \partial \theta_l} \Big|_{\boldsymbol{\theta}=0}. \quad (25)$$

Using the relations

$$\frac{\partial s_m}{\partial \theta_{m'}} = \delta_{m,m'}, \quad \frac{\partial^2 s_m}{\partial \theta_{m_1} \partial \theta_{m_2}} = \delta_{m,m_1} \delta_{m,m_2}, \quad (26)$$

and so forth, one finds that the cumulants of  $\tilde{n}_k$  are related to factorial cumulants Eqs. (20) - (23) as

$$\langle \tilde{n}_k \rangle_{\text{c}} = \langle \tilde{n}_k \rangle_{\text{fc}}, \quad (27)$$

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2} \rangle_{\text{c}} = \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \rangle_{\text{fc}} + \langle \tilde{n}_{k_1+k_2} \rangle_{\text{fc}}, \quad (28)$$

$$\begin{aligned} \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \rangle_{\text{c}} &= \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \rangle_{\text{fc}} + \langle \tilde{n}_{k_1} \tilde{n}_{k_2+k_3} \rangle_{\text{fc}} + (\text{comb.}) \\ &\quad + \langle \tilde{n}_{k_1+k_2+k_3} \rangle_{\text{fc}}, \end{aligned} \quad (29)$$

$$\begin{aligned} \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \tilde{n}_{k_4} \rangle_{\text{c}} &= \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3} \tilde{n}_{k_4} \rangle_{\text{fc}} + \langle \tilde{n}_{k_1} \tilde{n}_{k_2} \tilde{n}_{k_3+k_4} \rangle_{\text{fc}} + (\text{comb.}) \\ &\quad + \langle \tilde{n}_{k_1+k_2} \tilde{n}_{k_3+k_4} \rangle_{\text{fc}} + (\text{comb.}) + \langle \tilde{n}_{k_1} \tilde{n}_{k_2+k_3+k_4} \rangle_{\text{fc}} + (\text{comb.}) \\ &\quad + \langle \tilde{n}_{k_1+k_2+k_3+k_4} \rangle_{\text{fc}}, \end{aligned} \quad (30)$$

where (comb.) means the sum for all possible combinations for subscripts of  $\tilde{n}_k$ ; for example,

$$\langle \tilde{n}_{k_1} \tilde{n}_{k_2+k_3} \rangle_{\text{fc}} + (\text{comb.}) = \langle \tilde{n}_{k_1} \tilde{n}_{k_2+k_3} \rangle_{\text{fc}} + \langle \tilde{n}_{k_2} \tilde{n}_{k_3+k_1} \rangle_{\text{fc}} + \langle \tilde{n}_{k_3} \tilde{n}_{k_1+k_2} \rangle_{\text{fc}}. \quad (31)$$

#### D. Continuum limit

Next, we take the continuum limit,  $a \rightarrow 0$ . First, we denote the spatial coordinate of the  $m$ th cell as  $x = ma$ . In the continuum limit, using the particle number density defined by  $n(x) = n_m/a$ , the probability distribution function in Eq. (1) is promoted to the functional  $P[n(x), t]$  of  $n(x)$ . Note, however, that this notation is conceptual; in actual applications, the functional  $P[n(\eta), \tau]$  is always understood as the limit of the function  $P(\mathbf{n}, \tau)$  with small but finite  $a$ . The Fourier transform of  $n(x)$  is  $\tilde{n}(p) = \int dx e^{-ipx} n(x)$  with  $p = 2\pi k/Na$ . One can easily verify that the factorial cumulants of  $\tilde{n}(p)$  in the continuum notation are obtained by replacing  $\tilde{n}_k \rightarrow \tilde{n}(p)$  in Eqs. (20) - (23) with

$$\omega_p(t) = \gamma(t) a^2 p^2. \quad (32)$$

We require that the deterministic part of Eq. (1) obeys the solution of the diffusion equation Eq. (2) in the continuum limit. The solution of Eq. (2) in Fourier space with the initial condition  $n(x, 0) = M(x)$  is

$$\tilde{n}(p, t) = \tilde{M}(p) \exp\left[-\int_0^t dt' D(t') p^2\right]. \quad (33)$$

By comparing Eq. (33) with Eq. (20), one finds that these equations give the same result when one takes

$$D(t) = \gamma(t) a^2. \quad (34)$$

The  $a \rightarrow 0$  limit therefore has to be taken with Eq. (34). Similarly, one can show that the time evolution of the second order cumulant in the DME, Eq. (21), in this limit agrees with the solution of the stochastic diffusion equation Eq. (3) [19].

### E. Particle number in a volume

Next, we investigate the cumulants of the particle number in a range with length  $\Delta$ ,

$$Q(\Delta, t) = \int_{-\Delta/2}^{\Delta/2} dx n(x, t). \quad (35)$$

To represent the cumulants of  $Q$ , we first note that the diffusion distance of each Brownian particle in the model is given by

$$d(t) = \sqrt{2 \int_0^t dt' D(t')}, \quad (36)$$

from Eq. (33) [21, 41]. Using this quantity having the dimension of length, one can define a dimensionless variable

$$X = \frac{d(t)}{\Delta}. \quad (37)$$

The  $t$  and  $\Delta$  dependences of  $\langle (Q(\Delta, t))^n \rangle_c / \Delta$  then are described only through this dimensionless variable, because  $\langle (Q(\Delta, t))^n \rangle_c / \Delta$  is dimensionless and Eq. (37) is the natural dimensionless variable which can be created from the combination of  $\Delta$ ,  $t$  and  $D(t)$ .

Dependences of the first order cumulant on  $\Delta$  and  $t$  is calculated to be

$$\begin{aligned} \langle Q(\Delta, t) \rangle_c &= \int_{-\Delta/2}^{\Delta/2} dx \langle n(x) \rangle_c = \int_{-\Delta/2}^{\Delta/2} dx \int \frac{dp}{2\pi} e^{ipx} \langle \tilde{n}(p) \rangle_c \\ &= \int_{-\Delta/2}^{\Delta/2} dx \int \frac{dp}{2\pi} e^{ipx} \tilde{M}(p) e^{-\omega_p t} \\ &= \int dz M(z) \int_{-\Delta/2}^{\Delta/2} dx \int \frac{dp}{2\pi} e^{ip(x-z)} e^{-Dtp^2} \\ &= \int dz M(z) I_X(z/\Delta). \end{aligned} \quad (38)$$

Similarly, the second order cumulant of  $Q(\Delta, t)$  is calculated to be

$$\begin{aligned} \langle (Q(\Delta, t))^2 \rangle_c &= \int_{-\Delta/2}^{\Delta/2} dx_1 dx_2 \int \frac{dp_1 dp_2}{(2\pi)^2} e^{i(p_1 x_1 + p_2 x_2)} \langle \tilde{n}(p_1) \tilde{n}(p_2) \rangle_c \\ &= \int_{-\Delta/2}^{\Delta/2} dx_1 dx_2 \int \frac{dp_1 dp_2}{(2\pi)^2} e^{i(p_1 x_1 + p_2 x_2)} \\ &\quad \times \tilde{M}(p_1 + p_2) (e^{-\omega_{p_1+p_2} t} - e^{-(\omega_{p_1} + \omega_{p_2}) t}) \\ &= \int dz M(z) \int_{-\Delta/2}^{\Delta/2} dx_1 dx_2 \int \frac{dp_1 dp_2}{(2\pi)^2} e^{i(p_1(x_1-z) + p_2(x_2-z))} \\ &\quad \times (e^{-Dt(p_1+p_2)^2} - e^{-Dt(p_1^2+p_2^2)}) \\ &= \int dz M(z) (I_X(z/\Delta) - I_X(z/\Delta)^2). \end{aligned} \quad (39)$$

To obtain the last line, it is convenient to use a relation

$$\int \frac{dp_1 dp_2}{(2\pi)^2} e^{ip_1 x_1} e^{ip_2 x_2} f(p_1 + p_2) = \delta(x_1 - x_2) \int \frac{dp}{2\pi} e^{ip x_1} f(p). \quad (40)$$

In the last line we have defined

$$I_X(\zeta) = \int_{-1/2}^{1/2} d\xi \int \frac{dp}{2\pi} e^{-X^2 p^2/2} e^{ip(\xi+\zeta)} = \frac{1}{2} \left( \operatorname{erf} \frac{\zeta+1/2}{\sqrt{2}X} - \operatorname{erf} \frac{\zeta-1/2}{\sqrt{2}X} \right), \quad (41)$$

where  $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x dt e^{-t^2}$  is the error function. Some properties of Eq. (41) is summarized in appendix C.

Repeating similar manipulations, one finds that higher order cumulants of  $Q(\Delta, t)$  up to fourth order are given by

$$\langle (Q(\Delta, t))^n \rangle_c = \int dz M(z) H_n(z), \quad (42)$$

with

$$H_1(z) = I_X(z/\Delta), \quad (43)$$

$$H_2(z) = I_X(z/\Delta) - I_X(z/\Delta)^2, \quad (44)$$

$$H_3(z) = I_X(z/\Delta) - 3I_X(z/\Delta)^2 + 2I_X(z/\Delta)^3, \quad (45)$$

$$H_4(z) = I_X(z/\Delta) - 7I_X(z/\Delta)^2 + 12I_X(z/\Delta)^3 - 6I_X(z/\Delta)^4. \quad (46)$$

## F. Uniform initial condition

As a special case of the above result, it is instructive to consider the solution of the DME for a fixed initial condition with a uniform density,  $n(x, 0) = M(x) = M$ . With this initial condition, the average density trivially takes a constant value for all  $t$ . The fluctuations, however, have nontrivial  $t$  dependence even for this simple case, because they increase toward the equilibrated value from zero as  $t$  becomes larger.

Substituting the initial condition  $M(x) = M$  in Eq. (42) one obtains

$$\langle Q(\Delta, t) \rangle_c = M\Delta, \quad (47)$$

$$\langle (Q(\Delta, t))^2 \rangle_c = M\Delta (1 - F_2(X)), \quad (48)$$

$$\langle (Q(\Delta, t))^3 \rangle_c = M\Delta (1 - 3F_2(X) + 2F_3(X)), \quad (49)$$

$$\langle (Q(\Delta, t))^4 \rangle_c = M\Delta (1 - 7F_2(X) + 12F_3(X) - 6F_4(X)), \quad (50)$$

with

$$F_n(X) = \int dz [I_X(z/\Delta)]^n. \quad (51)$$

Some properties of  $F_n(X)$  are given in Appendix C. Note that the  $\Delta$  and  $t$  dependences of the cumulants are encoded through  $X$ ; as in Eq. (37),  $X = 0$  corresponds to  $t = 0$  while  $X$  goes to infinity for  $t \rightarrow \infty$  with a nonzero  $\Delta$ .

Eq. (47) shows that the average  $\langle Q \rangle = \langle Q \rangle_c = M\Delta$  does not have  $t$  dependence. On the other hand, the second and higher order cumulants, Eqs. (48) - (50), are  $t$  dependent. As discussed in Appendix C,  $F_n(X)$  for  $n \geq 2$  are monotonically decreasing functions of  $X$  with  $F_n(0) = 1$  and  $\lim_{X \rightarrow \infty} F_n(X) = 0$ . From these properties, one finds that Eqs. (48) - (50) vanish at  $t = 0$  as are required by the fixed initial condition, while they approach the Poissonian value  $\langle (Q(\Delta, t))^n \rangle_c = M\Delta$  in the large  $t$  limit.

In Fig. 2, we show the second, third and fourth order cumulants normalized by their equilibrated value,  $M\Delta$ , as functions of  $X$  and  $1/X$ . Because  $X$  is a monotonically increasing function of  $t$  with fixed  $\Delta$ , the left panel can be seen as the time evolution of  $\langle (Q(\Delta, t))^n \rangle_c$ . The panel shows that the second and third order cumulants are monotonically increasing functions. The fourth order one, on the other hand, has a non monotonic time evolution; it first becomes negative, and then increases toward the equilibrated value. It is worth emphasizing that the fourth order cumulant can become negative owing to the non-equilibrium effect. One also finds from the figure that the approach of the cumulant to the equilibrated value is slower for higher order. An intuitive interpretation of this tendency is discussed in Appendix B.



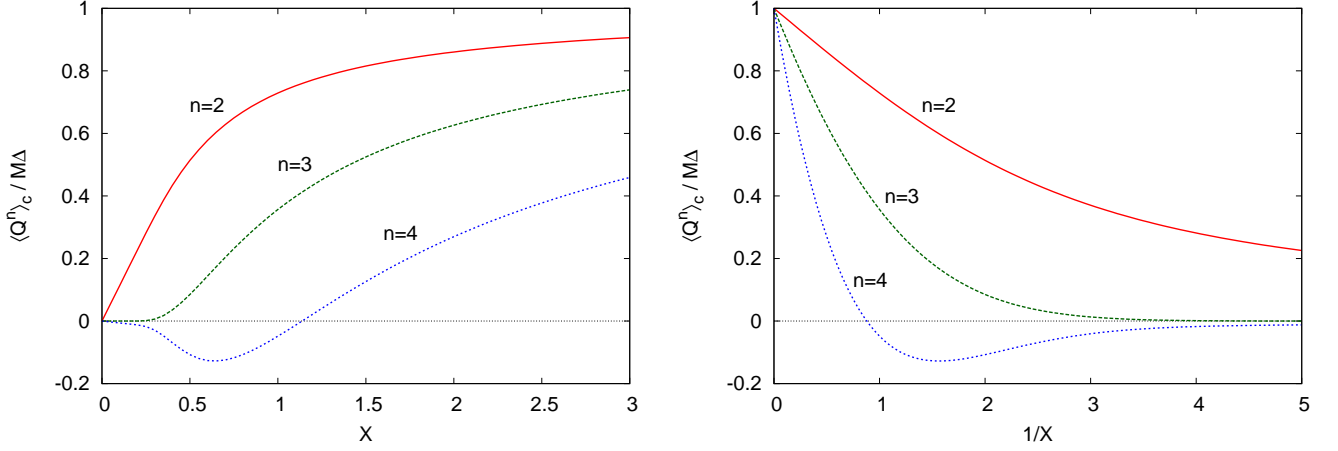


FIG. 2. Dependences of the cumulants of  $Q(\Delta, \tau)$  on  $X = d(t)/\Delta$  (left) and  $1/X$  (right).

In the right panel of Fig. 2, the same quantities are shown as functions of  $1/X = \Delta/d(t)$ . With fixed  $t$ , this plot can be regarded as the  $\Delta$  dependence of  $\langle (Q(\Delta, t))^n \rangle_c / M\Delta$ . The figure shows that the equilibration of  $\langle (Q(\Delta, t))^n \rangle_c$  is realized in the  $\Delta \rightarrow 0$  limit, while  $\langle (Q(\Delta, t))^n \rangle_c$  approaches the initial value as  $\Delta$  becomes larger. In other words, the approach to the equilibrated value is faster for smaller  $\Delta$ . We note that such a behavior of second-order cumulant of net-electric charge fluctuation is observed by the ALICE collaboration [5], as will be discussed in more detail in Sec. IV C.

### G. General solution

So far, we have considered the DME Eq. (1) for the fixed initial conditions,  $n(x, 0) = M(x)$  without fluctuations. Now, we extend these results to general initial conditions having fluctuations. This procedure is nicely carried out by a superposition of the previous results and using the formula of cumulants given in appendix A.

Let us first denote the probability distribution in the initial condition at  $t = 0$  as  $P[n(x), 0] = F[n(x)]$ . Because Eq. (1) is a linear equation, probability distribution  $P[n(x), t]$  with the initial distribution  $P[n(x), 0] = F[n(x)]$  is then given by

$$P[n(x), t] = \sum_{\{M(x)\}} F[M(x)] P_{M(x)}[n(x), t], \quad (52)$$

where  $P_{M(x)}[n(x), t]$  is the probability distribution function at time  $t$  with the fixed initial condition with  $n(z, 0) = M(x)$ , and the sum runs over the function space of  $M(x)$ .

Eq. (52) shows that  $P[n(x), t]$  is given by the superposition of probabilities  $P_{M(x)}[n(x), t]$  with an weight  $F[M(x)]$ . In addition, the cumulants of  $P_{M(x)}[n(x), t]$ , given in Eq. (42), are linear with respect to  $M(x)$ . Therefore, the

cumulants of Eq. (52) is obtained by substituting Eq. (42) to Eqs. (A20) - (A23), which results in

$$\langle Q(\Delta, t) \rangle_c = \int dz \langle M(z) \rangle_{c,0} H_1(z), \quad (53)$$

$$\begin{aligned} \langle (Q(\Delta, t))^2 \rangle_c &= \int dz_1 dz_2 \langle M(z_1) M(z_2) \rangle_{c,0} H_1(z_1) H_1(z_2) \\ &\quad + \int dz \langle M(z) \rangle_{c,0} H_2(z), \end{aligned} \quad (54)$$

$$\begin{aligned} \langle (Q(\Delta, t))^3 \rangle_c &= \int dz_1 dz_2 dz_3 \langle M(z_1) M(z_2) M(z_3) \rangle_{c,0} H_1(z_1) H_1(z_2) H_1(z_3) \\ &\quad + 3 \int dz_1 dz_2 \langle M(z_1) M(z_2) \rangle_{c,0} H_1(z_1) H_2(z_2) \\ &\quad + \int dz \langle M(z) \rangle_{c,0} H_3(z), \end{aligned} \quad (55)$$

$$\begin{aligned} \langle (Q(\Delta, t))^4 \rangle_c &= \int dz_1 dz_2 dz_3 dz_4 \langle M(z_1) M(z_2) M(z_3) M(z_4) \rangle_{c,0} \prod_{i=1}^4 H_1(z_i) \\ &\quad + 6 \int dz_1 dz_2 dz_3 \langle M(z_1) M(z_2) M(z_3) \rangle_{c,0} H_1(z_1) H_1(z_2) H_2(z_3) \\ &\quad + \int dz_1 dz_2 \langle M(z_1) M(z_2) \rangle_{c,0} \{3H_2(z_1) H_2(z_2) + 4H_1(z_1) H_3(z_2)\} \\ &\quad + \int dz \langle M(z) \rangle_{c,0} H_4(z). \end{aligned} \quad (56)$$

where  $\langle M(z_1) M(z_2) \cdots \rangle_{c,0}$  represents the cumulants of the initial distribution  $F[M(z)]$ . Note that the above results up to second order are consistent with the solution of the stochastic diffusion equation.

## H. Initial condition satisfying uniformity and locality

Using Eqs. (53) - (56) one can obtain cumulants of  $Q(\Delta, t)$  with an arbitrary initial condition having fluctuations. We next constrain the initial condition so that the result becomes physically apparent.

In an equilibrated medium and in a macroscopic scale, density fluctuations are local, i.e. the cumulants of the particle density is given by

$$\langle n(x_1) n(x_2) \cdots n(x_l) \rangle_c = [n^l]_c \delta(x_1 - x_2) \cdots \delta(x_1 - x_l), \quad (57)$$

where the coefficients in Eq. (57) defined by

$$[n^l]_c = \langle Q^n \rangle_c / \Delta, \quad (58)$$

represent the generalized susceptibility. This is because the correlation length of the density would be at most the same order as the microscopic scales such as the mean free path without long range correlations. The correlation between macroscopically separated points, therefore, must vanishes. It is notable that Eq. (57) is also satisfied in the classical free gas in grand canonical ensemble. When, and only when, the cumulants satisfy the condition Eq. (57),  $\langle Q^n \rangle_c$  are extensive variables and proportional to  $\Delta$ ,

$$\langle (Q(\Delta, t))^n \rangle_c = \int_0^\Delta dx_1 \cdots dx_n \langle n(x_1) n(x_2) \cdots n(x_l) \rangle_{c,0} = \Delta [n^l]_c. \quad (59)$$

It is reasonable from Eq. (59) to refer to  $[n^l]_c$  as the generalized susceptibility. Note that the translational symmetry of the system is assumed in Eq. (57); extension of Eq. (57) to the finite volume system is addressed in Ref. [21].

For initial conditions satisfying Eq. (57), the cumulants Eqs. (53) - (56) are reduced to be

$$\langle Q(\Delta, t) \rangle_c = \Delta[M]_c, \quad (60)$$

$$\langle (Q(\Delta, t))^2 \rangle_c = \Delta[M]_c \{1 - F_2(X)\} + \Delta[M^2]_c F_2(X), \quad (61)$$

$$\begin{aligned} \langle (Q(\Delta, t))^3 \rangle_c = & \Delta[M]_c \{1 - 3F_2(X) + 2F_3(X)\} + 3\Delta[M^2]_c \{F_2(X) - F_3(X)\} \\ & + \Delta[M^3]_c F_3(X), \end{aligned} \quad (62)$$

$$\begin{aligned} \langle (Q(\Delta, t))^4 \rangle_c = & \Delta[M]_c \{1 - 7F_2(X) + 12F_3(X) - 6F_4(X)\} \\ & + \Delta[M^2]_c \{7F_2(X) - 18F_3(X) + 11F_4(X)\} \\ & + 6\Delta[M^3]_c \{F_3(X) - F_4(X)\} + \Delta[M^4]_c F_4(X), \end{aligned} \quad (63)$$

where  $[M^n]_c$  are the generalized susceptibilities for the initial condition satisfying

$$[M^n]_c = \frac{1}{\Delta} \left\langle \left( \int_0^\Delta dz M(z) \right)^n \right\rangle. \quad (64)$$

### III. EXTENSION TO MULTI-PARTICLE SYSTEMS

#### A. Cumulants of net numbers

Next, let us extend our model to a system composed of two species of Brownian particles. We then consider the cumulants of the difference of the particle numbers. This extension is needed for the description the cumulants of the net-baryon and net-electric charge numbers, which are defined by the difference of particle numbers.

We denote the densities of two particle species as  $n_1(x)$  and  $n_2(x)$ , and assume that their time evolutions are given by Eq. (1). We then consider the cumulants of the difference of the particle numbers in a range of width  $\Delta$ ,

$$Q_{\text{net}}(\Delta, t) = Q_1(\Delta, t) - Q_2(\Delta, t), \quad (65)$$

$$Q_i(\Delta, t) = \int_{-\Delta/2}^{\Delta/2} dx n_i(x, t). \quad (66)$$

One easily finds that the distribution of  $Q_{\text{net}}$  approaches the Skellam one in the  $t \rightarrow \infty$  limit, because in this limit the distributions of  $Q_1$  and  $Q_2$  are given by Poissonian and uncorrelated.

For clarity now we recover the discretized notation for the particle density for the moment. We write the probability distribution function that a cell labeled by  $m$  have  $n_m^{(1)}$  and  $n_m^{(2)}$  particles at time  $t$  as  $P(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, t)$ . Because  $n_m^{(1)}$  and  $n_m^{(2)}$  separately obey Eq. (1), if these two densities have no correlation in the initial condition,  $n_m^{(1)}$  and  $n_m^{(2)}$  are independent for any  $t$ . In this case, the probability can be factorized as

$$P(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, t) = P_1(\mathbf{n}^{(1)}, t) P_2(\mathbf{n}^{(2)}, t). \quad (67)$$

When  $P(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, t)$  is factorized in this way, the cumulant generating function  $K(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, t)$  defined by

$$K(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, t) = \sum_{\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}} \prod_m e^{\boldsymbol{\theta}_m^{(1)} n_m^{(1)} + \boldsymbol{\theta}_m^{(2)} n_m^{(2)}} P(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, t), \quad (68)$$

is also decomposed as

$$K(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, t) = K_1(\boldsymbol{\theta}_1, t) + K_2(\boldsymbol{\theta}_2, t), \quad (69)$$

where  $K(\boldsymbol{\theta}, t)$  is the cumulant generating function for the distribution function of each particle.

Using the generating function Eq. (68), cumulants of the difference of particle numbers  $\bar{n}_m = n_m^{(1)} - n_m^{(2)}$  are given by

$$\langle \bar{n}_{m_1} \bar{n}_{m_2} \cdots \bar{n}_{m_l} \rangle_c = \frac{\partial^l K}{\partial \bar{\theta}_{m_1} \partial \bar{\theta}_{m_2} \cdots \partial \bar{\theta}_{m_l}}, \quad (70)$$

where  $\partial/\partial\bar{\theta} = \partial/\partial\bar{\theta}_1 - \partial/\partial\bar{\theta}_2$ . By combining Eqs. (70) and (69), one finds

$$\langle \bar{n}_{m_1} \bar{n}_{m_2} \cdots \bar{n}_{m_l} \rangle_c = \langle n_{m_1}^{(1)} n_{m_2}^{(1)} \cdots n_{m_l}^{(1)} \rangle_c + (-1)^l \langle n_{m_1}^{(2)} n_{m_2}^{(2)} \cdots n_{m_l}^{(2)} \rangle_c. \quad (71)$$

In the continuum notation and after taking the integral over the range  $\Delta$ , one obtains

$$\langle (Q_{\text{net}}(\Delta, t))^n \rangle_c = \langle (Q_1(\Delta, t))^n \rangle_c + (-1)^n \langle (Q_2(\Delta, t))^n \rangle_c, \quad (72)$$

for initial conditions without correlation between two particle species.

When the fluctuations of  $n_1(x)$  and  $n_2(x)$  are correlated in the initial condition, the probability distribution is no longer factorized as in Eq. (67). In this case, we denote the probability distribution as a superposition of the solutions of fixed initial conditions,

$$\begin{aligned} P[n_1(x), n_2(x), t] &= \sum_{\{M_1(x), M_2(x)\}} F[M_1(z), M_2(z)] P_{M_1(z), M_2(z)}[n_1(x), n_2(x), t] \\ &= \sum_{\{M_1(x), M_2(x)\}} F[M_1(z), M_2(z)] P_{M_1(z)}[n_1(x), t] P_{M_2(z)}[n_2(x), t], \end{aligned} \quad (73)$$

where  $P_{M_1(z), M_2(z)}[n_1(x), n_2(x), t]$  are the solutions for the fixed initial condition with  $n_1(x, 0) = M_1(x)$  and  $n_2(x, 0) = M_2(x)$ . In the second line of Eq. (73) we used the fact that  $n_1(x)$  and  $n_2(x)$  are uncorrelated with the fixed initial condition. It is also concluded from this uncorrelated nature that the cumulants of  $P_{M_1(z), M_2(z)}[n_1(x), n_2(x), t]$  satisfy Eq. (72). Using Eq. (72) and the relation for superposition of probabilities in appendix A, the cumulants of  $Q_{\text{net}}(\Delta, t)$  are given by,

$$\langle Q_{\text{net}}(\Delta, t) \rangle_c = \int dz \langle M_{\text{net}}(z) \rangle_{c,0} H_1(z), \quad (74)$$

$$\begin{aligned} \langle (Q_{\text{net}}(\Delta, t))^2 \rangle_c &= \int dz_1 dz_2 \langle M_{\text{net}}(z_1) M_{\text{net}}(z_2) \rangle_{c,0} H_1(z_1) H_1(z_2) \\ &\quad + \int dz \langle M_{\text{tot}}(z) \rangle_{c,0} H_2(z), \end{aligned} \quad (75)$$

$$\begin{aligned} \langle (Q_{\text{net}}(\Delta, t))^3 \rangle_c &= \int dz_1 dz_2 dz_3 \langle M_{\text{net}}(z_1) M_{\text{net}}(z_2) M_{\text{net}}(z_3) \rangle_{c,0} H_1(z_1) H_1(z_2) H_1(z_3) \\ &\quad + 3 \int dz_1 dz_2 \langle M_{\text{net}}(z_1) M_{\text{tot}}(z_2) \rangle_{c,0} H_1(z_1) H_2(z_2) \\ &\quad + \int dz \langle M_{\text{net}}(z) \rangle_{c,0} H_3(z), \end{aligned} \quad (76)$$

$$\begin{aligned} \langle (Q_{\text{net}}(\Delta, t))^4 \rangle_c &= \int dz_1 dz_2 dz_3 dz_4 \langle M_{\text{net}}(z_1) M_{\text{net}}(z_2) M_{\text{net}}(z_3) M_{\text{net}}(z_4) \rangle_{c,0} \prod_{i=1}^4 H_1(z_i) \\ &\quad + 6 \int dz_1 dz_2 dz_3 \langle M_{\text{net}}(z_1) M_{\text{net}}(z_2) M_{\text{tot}}(z_3) \rangle_{c,0} H_1(z_1) H_1(z_2) H_2(z_3) \\ &\quad + 3 \int dz_1 dz_2 \langle M_{\text{tot}}(z_1) M_{\text{tot}}(z_2) \rangle_{c,0} H_2(z_1) H_2(z_2) \\ &\quad + 4 \int dz_1 dz_2 \langle M_{\text{net}}(z_1) M_{\text{net}}(z_2) \rangle_{c,0} H_1(z_1) H_3(z_2) \\ &\quad + \int dz \langle M_{\text{tot}}(z) \rangle_{c,0} H_4(z), \end{aligned} \quad (77)$$

with  $M_{\text{net}}(z) = M_1(z) - M_2(z)$  and  $M_{\text{tot}}(z) = M_1(z) + M_2(z)$ .

When the initial condition satisfies the locality condition Eq. (57), cumulants of  $Q_{\text{net}}(\Delta, t)$  are

$$\langle Q_{\text{net}}(\Delta, t) \rangle_c = \Delta [M_{\text{net}}]_c, \quad (78)$$

$$\langle (Q_{\text{net}}(\Delta, t))^2 \rangle_c = \Delta [M_{\text{tot}}]_c \{1 - F_2(X)\} + \Delta [M_{\text{net}}^2]_c F_2(X), \quad (79)$$

$$\begin{aligned} \langle (Q_{\text{net}}(\Delta, t))^3 \rangle_c = & \Delta [M_{\text{net}}]_c \{1 - 3F_2(X) + 2F_3(X)\} + 3\Delta [M_{\text{net}} M_{\text{tot}}]_c \{F_2(X) - F_3(X)\} \\ & + \Delta [M_{\text{net}}^3]_c F_3(X), \end{aligned} \quad (80)$$

$$\begin{aligned} \langle (Q_{\text{net}}(\Delta, t))^4 \rangle_c = & \Delta [M_{\text{tot}}]_c \{1 - 7F_2(X) + 12F_3(X) - 6F_4(X)\} \\ & + 3\Delta [(M_{\text{tot}})^2]_c \{F_2(X) - 2F_3(X) + F_4(X)\} \\ & + 4\Delta [M_{\text{net}}^2]_c \{F_2(X) - 3F_3(X) + 2F_4(X)\} \\ & + 6\Delta [M_{\text{net}}^2 M^{\text{tot}}]_c \{F_3(X) - F_4(X)\} + \Delta [M_{\text{net}}^4]_c F_4(X). \end{aligned} \quad (81)$$

Here,  $[(M_{\text{net}})^i (M_{\text{tot}})^j]_c$  are generalized susceptibilities defined similarly to Eq. (58).

The cumulants in equilibrium in this system is obtained by taking the large  $t$  limit in Eqs. (78) - (81). By taking  $X \rightarrow \infty$ ,  $F_n(X)$  for  $n \geq 2$  vanish and the cumulants of  $Q_{\text{net}}$  become the Skellam ones with

$$\langle (Q_{\text{net}})^{2i+1} \rangle_c = \Delta [M_{\text{net}}]_c, \quad \langle (Q_{\text{net}})^{2i} \rangle_c = \Delta [M_{\text{tot}}]_c, \quad (82)$$

with  $i$  being an integer.

### B. Generalization to multi-particle systems

The above result would be suitable to describe the time evolution of net-baryon number cumulants in the hadronic medium in heavy ion collisions, because the net-baryon number,  $N_{\text{B}}^{\text{net}}$ , is given by the difference of baryon and anti-baryon numbers,  $N_{\text{B}}$  and  $N_{\bar{\text{B}}}$ , respectively, as

$$N_{\text{B}}^{\text{net}} = N_{\text{B}} - N_{\bar{\text{B}}}. \quad (83)$$

When one considers the cumulants of net-electric charge, there are hadrons having  $\pm 2$  charge in the unit of elementary charge in addition to those having  $\pm 1$ . The net-electric charge is thus given by

$$N_{\text{Q}}^{\text{net}} = 2N_{\text{Q},2} + N_{\text{Q},1} - N_{\text{Q},-1} - 2N_{\text{Q},-2}, \quad (84)$$

where  $N_{\text{Q},n}$  is the number of hadrons with  $n$  electric charge. Owing to the existence of nonzero  $N_{\text{Q},\pm 2}$ , then, even in the HRG model in which all  $N_{\text{Q},n}$  are given by the Poisson distribution and uncorrelated, the distribution of  $N_{\text{Q}}^{\text{net}}$  deviates from the Skellam one.

To describe this non-Skellam property, one may extend the DME to include four particle species. By constructing the net-electric charge as in Eq. (84), the distribution of the net-electric charge number in equilibrium has a deviation from the Skellam one. The effects of the charge 2 hadrons, however, are not large in the HRG model because of their small abundance [13]. We thus do not consider their effects in this paper and left the incorporation of these effects for future study.

## IV. RAPIDITY WINDOW DEPENDENCE OF CUMULANTS IN HEAVY ION COLLISIONS

Next, using the result obtained in the previous section we consider the time evolution of the cumulants of conserved charges and their  $\Delta\eta$  dependence observed in relativistic heavy ion collisions. As for conserved charges, we consider the net-electric charge and net-baryon number, which are both observable in these experiments. Note that the measurement of the net-baryon number cumulants is possible [16, 17], although the detectors cannot observe neutral baryons.

In QCD at sufficiently low temperature and low baryon chemical potential, the thermodynamic quantities including cumulants of conserved charges are well described by the HRG model. In this model, the distributions of net-baryon number and net-electric charge are given by the Skellam distribution to a good approximation. In dense and/or hot medium, the cumulants of conserved charges are expected to have deviations from the HRG values reflecting the change of the medium property including phase transitions to the deconfined medium. For example, fluctuations of net-baryon number and net-electric charge normalized by entropy density or net-baryon number are suppressed in the deconfined phase reflecting the decrease of the charges carried by quasi-particle excitations [8–10]. Near the QCD

critical point, the cumulants are expected to show characteristic suppressions and enhancements reflecting the critical phenomena [11, 12, 15]. The goal of the experimental study of the cumulants of conserved charges is to find and confirm these signals [6].

These signals, which would be developed in the deconfined medium or near the QCD critical point, however, are not directly observable. Before they are observed by the detectors, the hot medium undergoes a time evolution in the hadronic stage. The fluctuations are modified in the hadronic medium, presumably toward the one of the equilibrated hadronic medium. When one interprets the experimental results on fluctuations, one must keep this effect in mind; direct comparisons of the experimental results on cumulants with a thermal value at some early stage in heavy ion collisions would lead to a wrong conclusion. The purpose of this section is to describe the diffusive process of fluctuations in the hadronic stage and to estimate their effects on experimental results using the solution of the DME obtained in previous sections.

In relativistic heavy ion collisions with sufficiently large collision energy per nucleon,  $\sqrt{s_{NN}}$ , the hot medium created by the collisions has an approximate boost invariance around mid-rapidity. Useful coordinates to describe such a system are the coordinate-space rapidity  $y = (1/2) \log(t+x)(t-x)$  and proper time  $\tau = \sqrt{t^2 - x^2}$ , where  $x$  denotes the longitudinal direction. In this coordinate, the diffusion equation in the Cartesian coordinate, Eq. (2), is rewritten as

$$\frac{\partial}{\partial \tau} n(\eta, \tau) = D_y(\tau) \frac{\partial^2}{\partial y^2} n(y, \tau), \quad (85)$$

where  $n(y, \tau)$  is the density per unit rapidity, and the diffusion constant in this coordinate,  $D_y(\tau)$ , is related to the Cartesian one  $D$  as

$$D_y(\tau) = D\tau^{-2}. \quad (86)$$

Because of Eq. (85), the diffusion constant  $D$  has to be replaced by  $D_y$  when one applies the result in the previous sections to the description of the diffusive process along the rapidity direction.

In experiments, the fluctuations of particle numbers in a pseudo-rapidity window  $\Delta\eta$  are observed after integrating out the transverse direction [2, 5]. The pseudo-rapidity window can be varied within the coverage of the detector. In the following, we regard the pseudo-rapidity window as the one of the coordinate space rapidity  $\eta$  [36]. The  $\Delta\eta$  dependence of the cumulants can then be compared with the  $\Delta$  dependence obtained in previous sections. The cumulants obtained in the previous section, Eqs. (78) - (81), depends on the initial condition and the diffusion distance  $d(t)$ . By fitting the experimental data by these results, one can constrain the initial condition and the value of  $d(t)$ , namely information on the thermodynamics and transport property of the hot medium.

In heavy ion collisions, the conserved charges in the total system is fixed and does not fluctuate. Because the hot medium created in heavy ion collisions is a finite volume system, the measurement of fluctuation in a sub-volume is affected by this effect. For sufficiently large  $\sqrt{s_{NN}}$  and moderate range of  $\Delta\eta$ , however, such an effect is suppressed because of the finite duration of the diffusive process in the hadronic medium [21].

### A. Initial condition

In this study, we set the medium at chemical freezeout time,  $\tau = \tau_0$ , as the initial condition, and apply the DME to describe the diffusive process in the hadronic medium until kinetic freezeout. For sufficiently large  $\sqrt{s_{NN}}$ , hadronization and chemical freezeout are expected to take place at almost the same time. Because the local density profile of conserved charges in each event is almost frozen within the short duration because of the charge conservation, if the fluctuations are well equilibrated in the early stage, the fluctuations of conserved charges at  $\tau = \tau_0$  will strongly reflect the thermal property in the deconfined medium. The fluctuations of non-conserving quantities, on the other hand, are not constrained by the conservation law, and thus would be sensitive to the hadronization mechanism rather than the primordial thermodynamics.

In this study, we assume that the locality condition Eq. (57) is satisfied at  $\tau = \tau_0$ . Note that this assumption is satisfied if the medium is fully equilibrated at this time. With this assumption, the initial condition is specified by the generalized susceptibilities. To describe the cumulants up to fourth order, we need six susceptibilities in the initial condition

$$[M_{\text{net}}^2]_c, \quad [M_{\text{net}}^3]_c, \quad [M_{\text{net}}^4]_c, \quad [M_{\text{net}} M_{\text{tot}}]_c, \quad [M_{\text{net}}^2 M_{\text{tot}}]_c, \quad [(M_{\text{tot}})^2]_c. \quad (87)$$

The first three susceptibilities are those of conserved charges. By normalizing them by their equilibrated values given in Eq. (82), we introduce the following three parameters

$$D_2 = [M_{\text{net}}^2]_c/[M_{\text{tot}}]_c, \quad D_3 = [M_{\text{net}}^3]_c/[M_{\text{net}}]_c, \quad D_4 = [M_{\text{net}}^4]_c/[M_{\text{tot}}]_c, \quad (88)$$

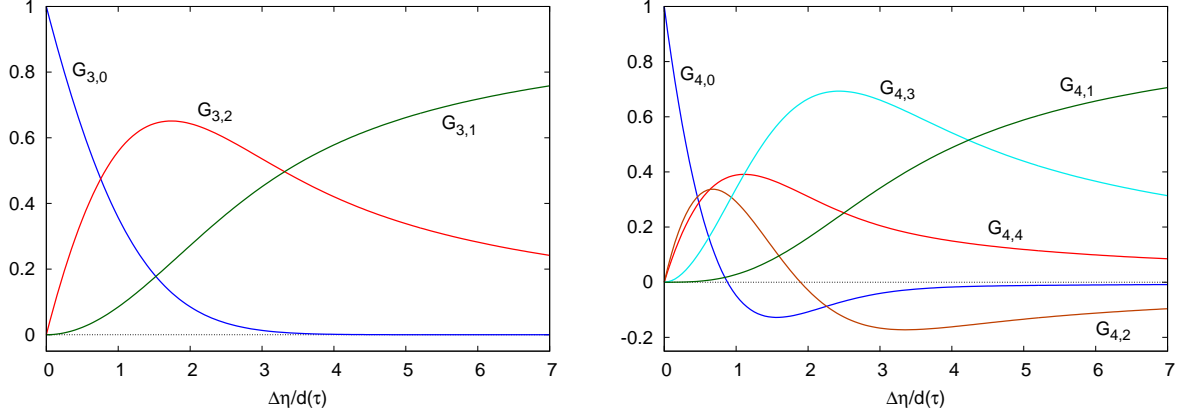


FIG. 3.  $\Delta\eta/d(\tau)$  dependences of the functions defined in Eqs. (96) - (99).

representing the magnitude of the cumulants of conserved charges at hadronization. Other three parameters are not the cumulants of conserved charges. We parametrize these quantities by

$$a = [M_{\text{net}}M_{\text{tot}}]_c/[M_{\text{net}}]_c, \quad b = [M_{\text{net}}^2M_{\text{tot}}]_c/[M_{\text{tot}}]_c, \quad c = [(M_{\text{tot}})^2]_c/[M_{\text{tot}}]_c, \quad (89)$$

where we again normalize them by the equilibrated values. Among these parameters,  $D_n$  are the quantities which should be compared with theoretical studies, such as lattice simulations, on the cumulants of conserved charges. Note that  $D_2$  and  $c$  are positive definite because they are second-order fluctuations, while other parameters can take both positive and negative values. In addition to Eqs. (88) and (89), the  $\Delta\eta$  dependence of the cumulants depends on the diffusion length  $d(\tau)$ .

Because the parameters in Eq. (89) are not directly constrained by the conservation law, their values would be sensitive to the hadronization mechanism rather than the primordial thermodynamics. In particular, the parameter  $c$  does not include the net-particle number and is completely free from the conservation law. Experimental constraint on this parameter would provide us with novel information on the hadronization mechanism [19].

For net-baryon number cumulants and for sufficiently low energy collisions at which the creation of the anti-baryons are well suppressed, the system can be regarded as the single particle one without anti-baryons. Because  $M_{\text{net}}$  and  $M_{\text{tot}}$  become identical in this case, one obtains  $a = c = D_2$  and  $b = D_3$ . The number of parameters are reduced in this case.

## B. Normalized cumulants

In the following, we plot the  $\Delta\eta$  dependences of cumulants by normalizing by their equilibrated values,

$$R_n(X) = \frac{\langle (Q_{\text{net}}(\Delta\eta, \tau))^n \rangle_c}{\lim_{\tau \rightarrow \infty} \langle (Q_{\text{net}}(\Delta\eta, \tau))^n \rangle_c}. \quad (90)$$

We refer to Eq. (90) as the normalized cumulants in what follows. Note that the equilibrated values are the Skellam one given in Eq. (82) in our model.

When one analyzes the normalized cumulants in experiments, one may use the equilibrated values in the HRG model for normalization. An alternative way to deduce the equilibrated value is to take the  $\Delta\eta \rightarrow 0$  limit of the experimental results on the cumulant.

Usually, the experimental results on the higher order cumulants are discussed in terms of their ratios such as  $R_4/R_2$  and  $R_3/R_1$  [2]. When one considers the  $\Delta\eta$  dependences, however, the normalized cumulants would be better than the ratios, since one can investigate the individual cumulants directly using  $R_n$ .

Using the parameters in Eqs. (88) and (89), the solution obtained in the previous section Eqs. (78) - (81) in terms of the normalized cumulants are given by

$$R_2(X) = G_{2,0}(X) + D_2 G_{2,1}(X), \quad (91)$$

$$R_3(X) = G_{3,0}(X) + D_3 G_{3,1}(X) + a G_{3,2}(X), \quad (92)$$

$$R_4(X) = G_{4,0}(X) + D_4 G_{4,1}(X) + D_2 G_{4,2}(X) + b G_{4,3}(X) + c G_{4,4}(X), \quad (93)$$

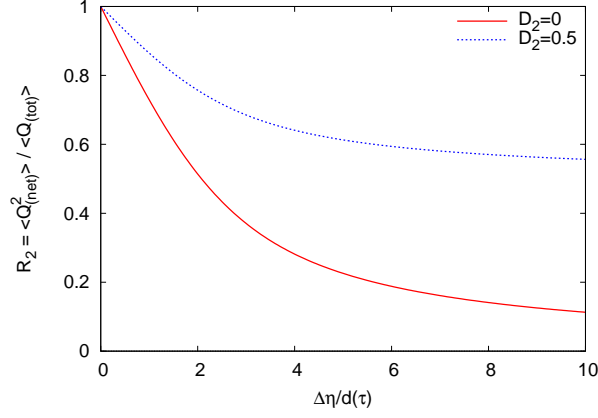


FIG. 4.  $\Delta\eta/d(\tau) = 1/X$  dependence of the normalized second order cumulant  $R_2$  for  $D_2 = 0$  and  $0.5$ .

where we have introduced the following functions:

$$G_{2,0}(X) = 1 - F_2(X), \quad G_{2,1}(X) = F_2(X), \quad (94)$$

$$G_{3,0}(X) = 1 - 3F_2(X) + 2F_3(X), \quad G_{3,1}(X) = F_3(X), \quad (95)$$

$$G_{3,2}(X) = 3(F_2(X) - F_3(X)), \quad (96)$$

$$G_{4,0}(X) = 1 - 7F_2(X) + 12F_3(X) - 6F_4(X), \quad G_{4,1}(X) = F_4(X), \quad (97)$$

$$G_{4,2}(X) = 4(F_2(X) - 3F_3(X) + 2F_4(X)), \quad (98)$$

$$G_{4,3}(X) = 6(F_3(X) - F_4(X)), \quad G_{4,4}(X) = 3(F_2(X) - 2F_3(X) + F_4(X)). \quad (99)$$

In Fig. 3, we show  $1/X = \Delta\eta/d(\tau)$  dependence of the functions in Eqs. (94) - (99). The figure shows that these functions behave differently as functions of  $\Delta\eta/d(\tau)$ . The structures of these functions are responsible for characteristic behaviors of  $\Delta\eta$  dependences of the cumulants discussed in the next subsections.

### C. Second order cumulant and diffusion distance

Now, let us examine  $\Delta\eta/d(\tau) = 1/X$  dependence of the normalized cumulants Eq. (90). We first consider the second order one  $R_2$ . As in Eq. (91),  $R_2$  depends on the initial condition only through  $D_2$ . In Fig. 4 we show the  $\Delta\eta/d(\tau)$  dependence of  $R_2$  for  $D_2 = 0$  and  $0.5$ .

Since we plot the  $\Delta\eta$  dependences of the cumulants as functions of  $\Delta\eta/d(\tau)$  throughout this section, it is instructive to give a rough estimate on the value of  $d(\tau)$  at this point by comparing Fig. 4 with the existing experimental results. The second order cumulant of net-electric charge has been observed by ALICE collaboration [5]. In Ref. [5], the result is plotted using the quantity called the  $D$ -measure [9], which is related to  $R_2$  as

$$D = 4R_2, \quad (100)$$

provided that  $Q_{\text{net}}$  is the net-electric charge. By comparing the result in Fig. 4 with Fig. 3 in Ref. [5], one can constrain the values of  $D_2$  and  $d(\tau)$  [21]. In Ref. [8], the value of  $D_2$  is estimated as  $D_2 = 0.5$ . Using this value of  $D_2$  and Fig. 3 in Ref. [5] one can estimate [21]

$$d(\tau) = 0.3 \sim 0.5. \quad (101)$$

This result shows that the maximum rapidity window of the ALICE detector,  $\Delta\eta_{\text{max}} = 1.6$  [5], corresponds to

$$\Delta\eta_{\text{max}}/d(\tau) = 3.2 \sim 5.3. \quad (102)$$

The ALICE detector thus can analyze  $\Delta\eta/d(\tau)$  dependences of cumulants for  $0 < \Delta\eta < \Delta\eta_{\text{max}}$ . Note that this estimate on  $d(\tau)$  strongly depends on the value of  $D_2$ ; for smaller value of  $D_2$ ,  $d(\tau)$  becomes much larger.

When one applies the net-baryon number to  $Q_{\text{net}}$ , the value of  $d(\tau)$  should be much smaller because the hadrons having baryon number, namely baryons, are considerably heavier than those having electric charge, which are dominated by pions in heavy ion collisions. This suggests that wider range of  $\Delta\eta/d(\tau)$  can be analyzed for net-baryon



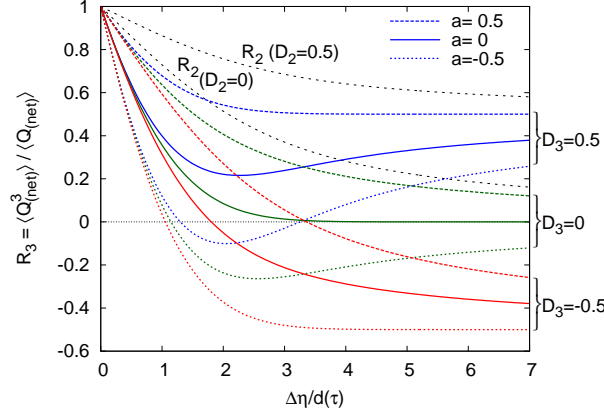


FIG. 5. Normalized third-order cumulants  $R_3$  as a function of  $1/X = \Delta\eta/d(\tau)$  for  $D_3 = -0.5, 0, 0.5$  and several values of  $a$ .

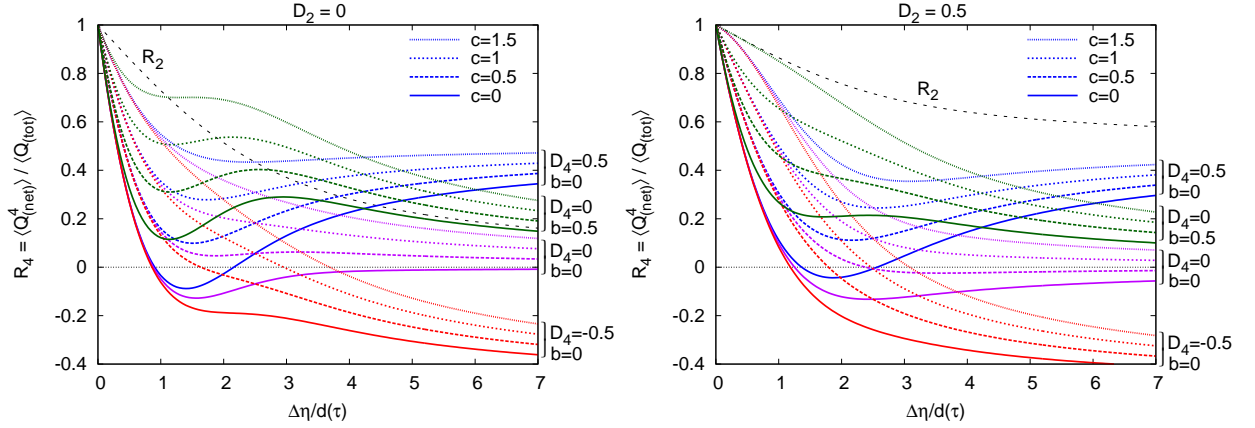


FIG. 6. Normalized fourth-order cumulants  $R_4$  as a function of  $\Delta\eta/d(\tau)$  for various initial parameters  $D_4$ ,  $D_2$ ,  $b$  and  $c$ . The left (right) panel shows the result for  $D_2 = 0$  ( $D_2 = 0.5$ ).

number cumulants with a fixed  $\Delta\eta$  coverage of an experimental detector. The use of the net-baryon number cumulants is advantageous compared with net-electric charge in this sense. The simultaneous measurements of both the net-baryon number and net-electric charge cumulants in a same experiment would provide us interesting results, such as the difference of the diffusion constants of net-baryon number and net-electric charge.

#### D. $\Delta\eta$ dependence of higher order cumulants

##### 1. Initial condition with small fluctuations

Next, we focus on the  $\Delta\eta$  dependences of third and fourth order cumulants. We first consider the initial condition with small fluctuations, since the fluctuations of net-baryon and net-electric charge numbers are expected to be suppressed in the deconfined medium [8–10]. The higher order cumulants can also be suppressed near the QCD critical point reflecting the critical phenomena [12, 15].

The normalized cumulant  $R_3$  depends on  $D_3$  and  $a$ , while  $R_4$  depends on  $D_4$ ,  $D_2$ ,  $b$  and  $c$ . In Fig. 5, we show dependence of  $R_3$  on  $\Delta\eta/d(\tau) = 1/X$  with  $D_3 = -0.5, 0$  and  $0.5$  for several values of  $a$ . In Fig. 6,  $\Delta\eta/d(\tau)$  dependence of  $R_4$  is shown for various values of the initial parameters. The figures show that the dependences of  $R_n$  on  $\Delta\eta/d(\tau)$  are sensitive to the initial parameters in the range  $\Delta\eta/d(\tau) \lesssim 7$ . For example, one sees from Fig. 5 that the behavior of  $R_3$  is sensitive to  $a$  for  $\Delta\eta/d(\tau) < 1$  while the effect of  $D_3$  becomes dominant as  $\Delta\eta/d(\tau)$  becomes larger. This behavior indicates that the experimental measurement of  $R_3$  in the range  $\Delta\eta/d(\tau) \lesssim 5$  makes it possible to constrain

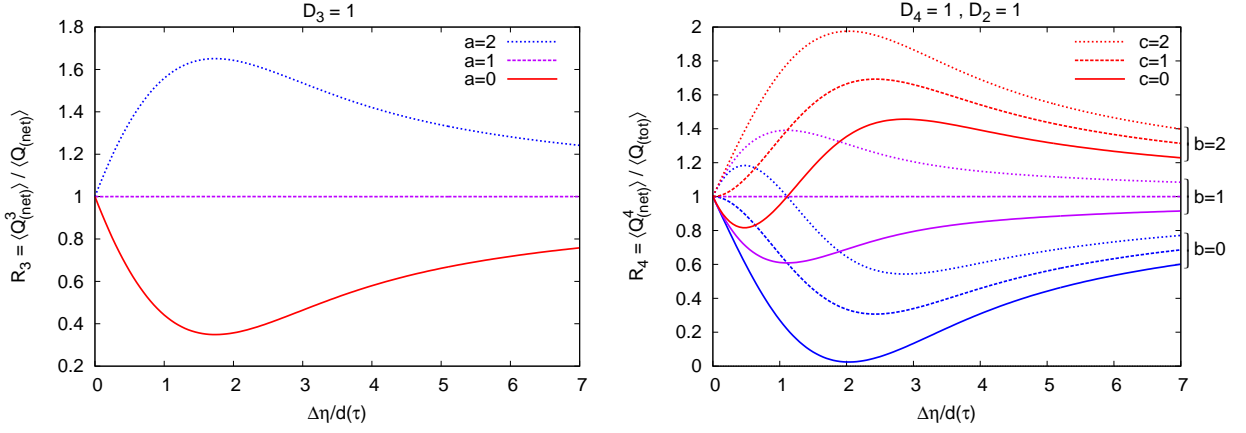


FIG. 7. Dependences of the normalized non-Gaussian cumulants  $R_3$  (left) and  $R_4$  (right) on  $\Delta\eta/d(\tau)$  with  $D_2 = D_3 = D_4 = 1$ .

two parameters  $D_3$  and  $a$ . It is also notable that  $\Delta\eta$  dependence of  $R_3$  can become non-monotonic for several choices of parameters. Experimental observation of such a non-monotonic behavior should be an interesting and unique signal to constrain these parameters and underlying physics. Figure 6 also shows that the behavior of  $R_4$  is sensitive to the initial parameters. For example,  $R_4$  can have two extrema for a choice of initial parameter in the range  $\Delta\eta/d(\tau) < 5$ . Although there are 16 lines for  $R_4$  in each panel of Fig. 6, these lines would be experimentally distinguished if the experimental result on  $R_4$  is obtained within 10% precision. This precision will be achieved in the BES-II program at RHIC [1]. Such experimental analysis can constrain the initial parameters  $D_4$ ,  $b$  and  $c$  as well as  $d(\tau)$ . We also note that, although  $R_4$  depends on  $D_2$ , the value of  $D_2$  can be constrained by the measurement of  $R_2$ . Similarly, because all normalized cumulants  $R_2$ ,  $R_3$  and  $R_4$  depend on a common  $d(\tau)$ , this parameter can be constrained by the simultaneous use of these three cumulants. In this way, the complementary use of these cumulants will lead us to a deeper understanding of the experimental results. The analyses of both the net-baryon and net-electric charge fluctuations in a same set of collision events will also give us further interesting information on the difference of the diffusion constants for these charges and so forth.

For the analysis of the net-baryon number cumulants with small  $\sqrt{s_{\text{NN}}}$ , because the abundance of anti-baryons are suppressed the parameters are constrained as  $a \simeq D_2$ ,  $c \simeq D_2$  and  $b \simeq D_3$ . Using these conditions, the analysis of the  $\Delta\eta$  dependence of normalized cumulants  $R_2$ ,  $R_3$  and  $R_4$  would further constrain the physics of fluctuations.

It is worth emphasizing that the results in Figs. 5 and 6 show that the value of a cumulant with a fixed  $\Delta\eta$  can deviate from both the initial and equilibrated values significantly. In particular, the sign of non-Gaussian cumulants can become negative even when the initial and equilibrated values are positive. These results show that the cumulants observed in experiments with a fixed  $\Delta\eta$  should not be compared with the theoretical analysis of the cumulants obtained in statistical mechanics assuming an equilibration. Total description of the  $\Delta\eta$  dependence of the cumulants are indispensable for proper understanding of the experimental results.

## 2. Initial condition with large cumulant

Next, we show the  $\Delta\eta$  dependence of the cumulants for initial conditions where higher order cumulants of conserved charges are consistent with or larger than the equilibrated values.

We first consider the case  $D_2 = D_3 = D_4 = 1$ , i.e. the cumulants of the conserved charge in the initial condition is given by the Skellam distribution. In Fig. 7, we show the  $\Delta\eta$  dependences of third and fourth order cumulants with this initial condition. In the figure, we vary the parameters  $a$ ,  $b$  and  $c$ , which are not constrained by the conservation law. The figure shows that the normalized cumulants for a given  $\Delta\eta$  can have a significant deviation from unity depending on the initial parameters although initial and equilibrated values of the cumulants are both unity. Only when  $a = b = c = 1$ , all normalized cumulants become unity irrespective of the value of  $\Delta\eta$ . This result again tells us that the measurement of the cumulants with a fixed  $\Delta\eta$  should not be compared with those in equilibrated medium.

Finally, we consider the  $\Delta\eta$  dependence of the normalized cumulants  $R_3$  and  $R_4$  for relatively large  $D_3$  and  $D_4$ . Such a case would be realized when the medium near the critical point is formed in the time evolution of the hot medium and the critical enhancement of fluctuations are well developed [11]. In Fig. 8 we show the  $\Delta\eta$  dependences of  $R_3$  and  $R_4$  with  $D_3 = 4$  and  $D_4 = 4$ , respectively, for several values of initial parameters. The figure tells us the

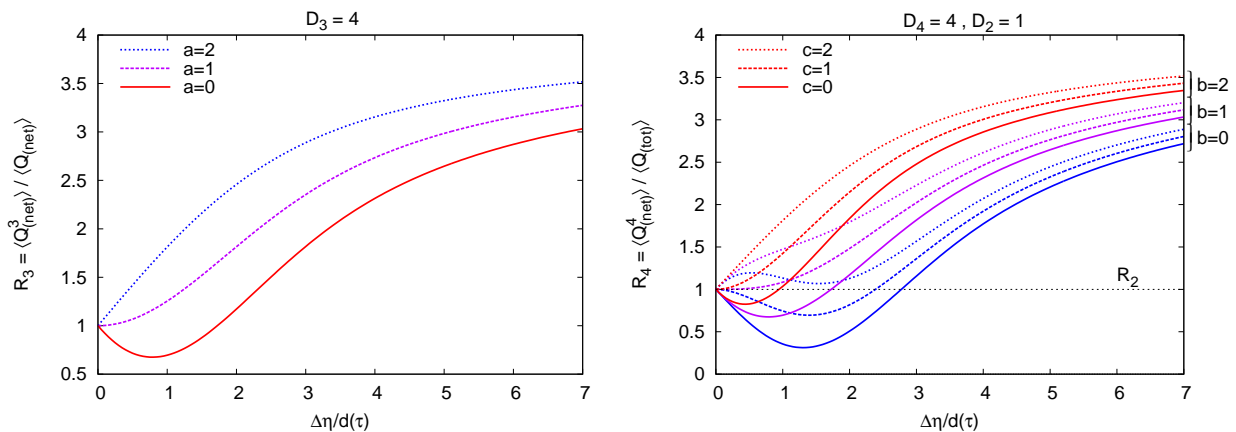


FIG. 8. Dependences of the normalized non-Gaussian cumulants  $R_3$  (left) and  $R_4$  (right) on  $\Delta\eta/d(\tau)$  with  $D_3 = 4$  and  $D_4 = 4$  with various initial parameters.

same conclusions as before: (1) The values of  $R_3$  and  $R_4$  with a fixed  $\Delta\eta$  can have significant deviation from the initial and equilibrated values, and (2) initial parameters can be constrained experimentally from the  $\Delta\eta$  dependences of the normalized cumulants.

## V. DISCUSSIONS AND SUMMARY

In the previous section we have discussed the rapidity window  $\Delta\eta$  dependence of the cumulants. These results can be directly compared with the experimental results of the cumulants of net-baryon number and net-electric charge. By this comparison, especially by the simultaneous use of the three normalized cumulants  $R_2$ ,  $R_3$  and  $R_4$ , parameters in the model and underlying physics on the thermal and transport properties of the hot medium can be constrained experimentally. The discussion in the previous section also tells us the importance of the total understanding of the  $\Delta\eta$  dependences of the experimental results. Because the value of the cumulants with a fixed  $\Delta\eta$  can deviate significantly from the initial and final values, they should not be directly compared with theoretical studies which are obtained in statistical mechanics assuming equilibration, such as lattice simulations. The quantities which should be compared with the cumulants in statistical mechanics are  $D_2$ ,  $D_3$  and  $D_4$ , which can be deduced by the analysis of the  $\Delta\eta$  dependence.

We note that the  $\Delta\eta$  dependence of the Gaussian fluctuation encodes the same physical information as those accessible with the balance function [42–44] of the corresponding particle species, because the former is obtained by the integral of the latter, and vice versa [21, 45, 46]. On the other hand, the  $\Delta\eta$  dependences of non-Gaussian cumulants are related to multi-particle correlation functions, and provide us with a novel information on the diffusive process of conserved charges.

All the results in this study is obtained using the DME. In this model it is assumed that the Brownian particles in the model do not interact with each other. This assumption would be qualitatively justified when  $Q_{\text{net}}$  is identified to be the net-baryon number because of the following two reasons. First, baryons in the hadronic medium mainly interact with pions, and the baryon-baryon interactions scarcely take place [17]. Second, the success of the thermal model for the particle abundances indicates that the pair annihilation of baryons is well suppressed after chemical freezeout [17]. The baryons thus can be regarded as the Brownian particles floating in the pionic medium without interacting with one another, and the correlation between two baryons in the hadronic medium will be well suppressed. For net-electric charges, on the other hand, the interactions, especially the pair-annihilation, occur frequently between hadrons having electric charge, especially pions. The present model thus would not be suitable for the quantitative description of the diffusive process of the net-electric charge fluctuations. It is desirable to extend the model to incorporate the interaction between particles for more quantitative description of the net-electric charge fluctuations. In the present study it is also assumed that the fluctuations satisfy the locality condition at chemical freezeout time, which, however, might be violated in realistic collision events. To incorporate such effects one has to modify the initial condition. We left these extensions to future study and do not address further in this study.

In this paper, we considered the time evolution and rapidity window,  $\Delta\eta$ , dependences of the cumulants of conserved charges observed in relativistic heavy ion collisions. In order to describe the non-equilibrium time evolution of non-

Gaussian cumulants we employed the diffusion master equation Eq. (1). The analytic formula for the time evolution of cumulants with arbitrary initial conditions are obtained. Using these results, we discussed the  $\Delta\eta$  dependence of cumulants of conserved charges observed in heavy ion collisions. Various suggestions have been made to utilize the  $\Delta\eta$  dependence of the non-Gaussian cumulants for the understanding of the thermal and transport properties of the hot medium.

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### Appendix A: Superposition of cumulants

In this appendix, we consider cumulants of a probability distribution function which is given by a superposition of probability distribution functions. Let us consider a probability distribution function  $P(x)$  for an integer stochastic variable  $x$ , and assume that  $P(x)$  consists of the superposition of sub-probabilities as

$$P(x) = \sum_N F(N) P_N(x), \quad (\text{A1})$$

where  $P_N(x)$  are sub-probabilities labeled by integer  $N$ . Each sub-probability is summed with a weight  $F(N)$  satisfying  $\sum_N F(N) = 1$ , which is also regarded as a probability. The purpose of this appendix is to represent the cumulants of  $P(x)$  using those of  $P_N(x)$  and  $F(N)$ . Although we write down the results explicitly up to fourth order in this manuscript, the result can be extended to higher orders straightforwardly. A similar result given in this appendix is presented and used in Ref. [17] to relate the net-proton and net-baryon number cumulants in the final state in relativistic heavy ion collisions.

We start from the cumulant generating function of Eq. (A1),

$$K(\theta) = \log \sum_x e^{\theta x} P(x) = \log \sum_N F(N) \sum_x e^{\theta x} P_N(x) \quad (\text{A2})$$

$$= \log \sum_N F(N) \sum_x e^{K_N(\theta)} \quad (\text{A3})$$

where  $K_N(\theta) = \log \sum_x e^{\theta x} P_N(x)$  is the cumulant generating function for  $P_N(x)$ . Using the cumulant expansion, Eq. (A3) is written as

$$K(\theta) = \sum_m \frac{1}{m!} \sum_F [K_N(\theta)]_c^m \quad (\text{A4})$$

$$= \sum_F K_N(\theta) + \frac{1}{2} \sum_F (\delta K_N(\theta))^2 + \frac{1}{3!} \sum_F (\delta K_N(\theta))^3 + \frac{1}{4!} \sum_F [K_N(\theta)]_c^4 + \dots, \quad (\text{A5})$$

where  $\sum_F$  is the shorthand notation of  $\sum_N F(N)$ , and  $\sum_F [K_N(\theta)]_c^m$  is the  $m$ th-order cumulant of  $K_N$  for the sum over  $F$ , whose explicit forms up to fourth order are given in the far right hand side with

$$\sum_F (\delta K_N(\theta))^n = \sum_F \left( K_N(\theta) - \sum_F K_N(\theta) \right)^n, \quad (\text{A6})$$

$$\sum_F [K_N(\theta)]_c^4 = \sum_F (\delta K_N(\theta))^4 - 3 \left( \sum_F (\delta K_N(\theta))^2 \right)^2. \quad (\text{A7})$$

Cumulants of  $P(x)$  are given by derivatives of  $K(\theta)$  as

$$\langle x^n \rangle_c = \frac{\partial^n}{\partial \theta^n} K(0) \equiv K^{(n)}. \quad (\text{A8})$$

All cumulants can be obtained with Eqs. (A8) and (A5). In order to calculate the cumulants explicitly, we first note that the normalization condition  $\sum_x P_N(x) = 1$  yields  $K_N(\theta) = 0$ . From this property, it is immediately concluded

that all  $K_N(\theta)$  in a term in the far right hand side of Eq. (A5) must receive at least one differentiation so that the term gives nonzero contribution to Eq. (A8). This means that the  $m$ th order term in Eq. (A5) can affect Eq. (A8) only if  $m \leq n$ . Keeping this rule in mind, derivatives of Eq. (A5) with  $\theta = 0$  is given by

$$K^{(1)} = \sum_F K_N^{(1)}, \quad (\text{A9})$$

$$K^{(2)} = \sum_F K_N^{(2)} + \sum_F (\delta K_N^{(1)})^2, \quad (\text{A10})$$

$$K^{(3)} = \sum_F K_N^{(3)} + 3 \sum_F \delta K_N^{(1)} \delta K_N^{(2)} + \sum_F (\delta K_N^{(1)})^3, \quad (\text{A11})$$

$$K^{(4)} = \sum_F K_N^{(4)} + 4 \sum_F \delta K_N^{(1)} \delta K_N^{(3)} + 3 \sum_F (\delta K_N^{(2)})^2 + 6 \sum_F (\delta K_N^{(1)})^2 \delta K_N^{(2)} + \sum_F (\delta K_N^{(1)})^4, \quad (\text{A12})$$

with  $K_N^{(n)} = \partial^n K_N(0)/\partial \theta^n$  being the cumulants of sub-probabilities  $P_N(x)$ . Equations (A9) - (A12) relate the cumulants  $K^{(n)}$  with  $K_N^{(n)}$ .

The above relations are further simplified when the cumulants of  $P_N(x)$  are at most linear with respect to  $N$ , i.e.

$$K_N^{(n)} = N \xi_{(n)} + \zeta_{(n)}, \quad (\text{A13})$$

where  $\xi_{(n)}$  and  $\zeta_{(n)}$  are constants which do not depend on  $N$ . Substituting Eq. (A13) in Eqs. (A9) - (A12) one obtains

$$K^{(1)} = \zeta_{(1)} + \xi_{(1)} \langle N \rangle_F, \quad (\text{A14})$$

$$K^{(2)} = \zeta_{(2)} + \xi_{(2)} \langle N \rangle_F + \xi_{(1)}^2 \langle \delta N^2 \rangle_F, \quad (\text{A15})$$

$$K^{(3)} = \zeta_{(3)} + \xi_{(3)} \langle N \rangle_F + 3 \xi_{(1)} \xi_{(2)} \langle \delta N^2 \rangle_F + \xi_{(1)}^3 \langle \delta N^3 \rangle_F, \quad (\text{A16})$$

$$K^{(4)} = \zeta_{(4)} + \xi_{(4)} \langle N \rangle_F + (4 \xi_{(1)} \xi_{(3)} + 3 \xi_{(2)}^2) \langle \delta N^2 \rangle_F + 6 \xi_{(1)}^2 \xi_{(2)} \langle \delta N^3 \rangle_F + \xi_{(1)}^4 \langle \delta N^4 \rangle_{c,F}, \quad (\text{A17})$$

where  $\langle O(N) \rangle_F = \sum_N O(N) F(N)$  denotes the average over  $F(N)$ ; these averages in the above formulas represent cumulants of the probability  $F(N)$ .

These results are easily generalized to the case where the sub-probabilities are labeled by multi-dimensional integers,

$$P(x) = \sum_{\mathbf{N}} F(\mathbf{N}) P_{\mathbf{N}}(x), \quad (\text{A18})$$

with  $\mathbf{N} = (N_1, N_2, \dots, N_m)$  representing a set of  $m$  integers. When the cumulants of  $P_{\mathbf{N}}(x)$  are linear in  $\mathbf{N}$ , i.e.,

$$K_{\mathbf{N}}^{(n)} = \mathbf{N} \cdot \boldsymbol{\xi}_{(n)} + \zeta_{(n)}, \quad (\text{A19})$$

a similar manipulation as before yields,

$$K^{(1)} = \zeta_{(1)} + \langle \boldsymbol{\xi}_{(1)} \cdot \mathbf{N} \rangle_F, \quad (\text{A20})$$

$$K^{(2)} = \zeta_{(2)} + \langle \boldsymbol{\xi}_{(2)} \cdot \mathbf{N} \rangle_F + \langle (\boldsymbol{\xi}_{(1)} \cdot \delta \mathbf{N})^2 \rangle_F, \quad (\text{A21})$$

$$K^{(3)} = \zeta_{(3)} + \langle \boldsymbol{\xi}_{(3)} \cdot \mathbf{N} \rangle_F + 3 \langle (\boldsymbol{\xi}_{(1)} \cdot \delta \mathbf{N})(\boldsymbol{\xi}_{(2)} \cdot \delta \mathbf{N}) \rangle_F + \langle (\boldsymbol{\xi}_{(1)} \cdot \delta \mathbf{N})^3 \rangle_F, \quad (\text{A22})$$

$$K^{(4)} = \zeta_{(4)} + \langle \boldsymbol{\xi}_{(4)} \cdot \mathbf{N} \rangle_F + 4 \langle (\boldsymbol{\xi}_{(1)} \cdot \delta \mathbf{N})(\boldsymbol{\xi}_{(3)} \cdot \delta \mathbf{N}) \rangle_F + 3 \langle (\boldsymbol{\xi}_{(2)} \cdot \delta \mathbf{N})^2 \rangle_F + 6 \langle (\boldsymbol{\xi}_{(1)} \cdot \delta \mathbf{N})^2 (\boldsymbol{\xi}_{(2)} \cdot \delta \mathbf{N}) \rangle_F + \langle (\boldsymbol{\xi}_{(1)} \cdot \delta \mathbf{N})^4 \rangle_{c,F}. \quad (\text{A23})$$

## Appendix B: Chemical Reaction

In this appendix, we consider a simple chemical reaction between two species of particles X and A [39],



It is assumed that a particle X (A) becomes A (X) with a probability  $k_1$  ( $k_2$ ) per unit time. We denote the number of each particle in the system as  $x$  and  $a$ . For simplicity, it is further assumed that  $a$  is sufficiently large compared with  $x$  so that the number  $a$  can be regarded fixed. The time evolution of the probability distribution  $P(x, t)$  is then described by the master equation [39]

$$\partial_t P(x, t) = k_2 a P(x-1, t) + k_1 (x+1) P(x+1, t) - (k_1 x + k_2 a) P(x, t). \quad (\text{B2})$$

In the following, we present the solution of this master equation for arbitrary initial conditions.

### 1. Fixed initial condition

Let us first solve Eq. (B2) with a fixed initial condition

$$P(x, 0) = \delta_{x, N}, \quad (\text{B3})$$

i.e. the number  $x$  is fixed to  $N$  without fluctuations at  $t = 0$ .

In order to solve Eq. (B2), we use factorial generating function,

$$G_f(s, t) = \sum_x s^x P(x, t). \quad (\text{B4})$$

Substituting Eq. (B4) in Eq. (B2), one obtains

$$\partial_t G_f(s, t) = k_2 a (s-1) G_f(s, t) - k_1 (s-1) \partial_s G_f(s, t). \quad (\text{B5})$$

Equation (B5) can be solved by using the method of characteristics, which yields

$$G_f(s, t) = F((s-1)e^{-k_1 t}) e^{(s-1)N_{\text{eq}}} \quad (\text{B6})$$

with an arbitrary function  $F$  and the average number of  $x$  in equilibrium,  $N_{\text{eq}} = k_2 a / k_1$ .

The function  $F$  is determined by specifying the initial condition. The generating function corresponding to the initial condition Eq. (B3) is  $G_f(s, 0) = s^N$ . This gives

$$F(r) = (r+1)^N e^{rN_{\text{eq}}}. \quad (\text{B7})$$

The solution of Eq. (B5) with the initial condition Eq. (B3) thus is

$$G_f(s, t) = (1 + (s-1)e^{-k_1 t})^N e^{N_{\text{eq}}(s-1)(1-e^{-k_1 t})}. \quad (\text{B8})$$

The moment and cumulant generating functions,  $G(\theta, t)$  and  $K(\theta, t)$ , respectively, are given by

$$G(\theta, t) = G_f(e^\theta, t), \quad (\text{B9})$$

$$K(\theta, t) = \log G(\theta, t). \quad (\text{B10})$$

Derivatives of  $K(\theta, t)$  define cumulants;  $\langle x(t)^n \rangle_c = \partial^n K(0, t) / \partial \theta^n$ . Using

$$\frac{\partial}{\partial \theta} G(\theta, t) = \frac{\partial s}{\partial \theta} \frac{\partial}{\partial s} G_f(s, t) = e^\theta \frac{\partial}{\partial s} G_f(s, t), \quad (\text{B11})$$

$$\frac{\partial^2}{\partial \theta^2} G(\theta, t) = e^\theta \frac{\partial}{\partial s} G_f(s, t) + e^{2\theta} \frac{\partial^2}{\partial s^2} G_f(s, t), \quad (\text{B12})$$

and so forth, one obtains

$$\langle x(t) \rangle_c = N_{\text{eq}}(1 - e^{-k_1 t}) + N e^{-k_1 t}, \quad (\text{B13})$$

$$\langle x(t)^2 \rangle_c = N_{\text{eq}}(1 - e^{-k_1 t}) + N(e^{-k_1 t} - e^{-2k_1 t}), \quad (\text{B14})$$

$$\langle x(t)^3 \rangle_c = N_{\text{eq}}(1 - e^{-k_1 t}) + N(e^{-k_1 t} - 3e^{-2k_1 t} + 2e^{-3k_1 t}), \quad (\text{B15})$$

$$\langle x(t)^4 \rangle_c = N_{\text{eq}}(1 - e^{-k_1 t}) + N(e^{-k_1 t} - 7e^{-2k_1 t} + 12e^{-3k_1 t} - 6e^{-4k_1 t}). \quad (\text{B16})$$

## 2. General initial condition

In order to obtain the  $t$  dependence of the cumulants for general initial conditions, we make use of the result in Appendix A. Since Eq. (B2) is a linear differential equation, the solution of  $P(x, t)$  for an initial condition  $P(x, 0) = F(x)$  is given by

$$P(x, t) = \sum_N F(N) P_N(x, t), \quad (\text{B17})$$

where  $P_N(x, t)$  is the solution for the fixed initial condition, Eq. (B3). We further remark that the cumulants Eqs. (B13) - (B16) are at most linear in  $N$  and fulfill the form in Eq. (A13) with

$$\zeta_{(n)} = N_{\text{eq}}(1 - e^{-k_1 t}), \quad \xi_{(1)} = e^{-k_1 t}, \quad \xi_{(2)} = e^{-k_1 t} - e^{-2k_1 t}, \quad (\text{B18})$$

and etc. The cumulants for arbitrary initial conditions thus are obtained by substituting Eqs. (B18) in Eqs. (A14) - (A17). The results up to fourth order are

$$\langle x(t) \rangle_c = N_{\text{eq}}(1 - e^{-k_1 t}) + \langle N \rangle_0 e^{-k_1 t}, \quad (\text{B19})$$

$$\langle x(t)^2 \rangle_c = N_{\text{eq}}(1 - e^{-k_1 t}) + \langle N \rangle_0 e^{-k_1 t} + (\langle \delta N^2 \rangle_0 - \langle N \rangle_0) e^{-2k_1 t}, \quad (\text{B20})$$

$$\begin{aligned} \langle x(t)^3 \rangle_c = & N_{\text{eq}}(1 - e^{-k_1 t}) + \langle N \rangle_0 e^{-k_1 t} + 3(\langle \delta N^2 \rangle_0 - \langle N \rangle_0) e^{-2k_1 t} \\ & + (\langle \delta N^3 \rangle_0 - 3\langle \delta N^2 \rangle_0 + 2\langle N \rangle_0) e^{-3k_1 t}, \end{aligned} \quad (\text{B21})$$

$$\begin{aligned} \langle x(t)^4 \rangle_c = & N_{\text{eq}}(1 - e^{-k_1 t}) + \langle N \rangle_0 e^{-k_1 t} + 7(\langle \delta N^2 \rangle_0 - \langle N \rangle_0) e^{-2k_1 t} \\ & + 6(\langle \delta N^3 \rangle_0 - 3\langle \delta N^2 \rangle_0 + 2\langle N \rangle_0) e^{-3k_1 t} \\ & + (\langle \delta N^4 \rangle_{c,0} - 6\langle \delta N^3 \rangle_0 + 11\langle \delta N^2 \rangle_0 - 6\langle N \rangle_0) e^{-4k_1 t}, \end{aligned} \quad (\text{B22})$$

where  $\langle O(N) \rangle_0$  denotes the average in the initial condition.

From Eqs. (B19) - (B22), one finds that in the large  $t$  limit,  $k_1 t \rightarrow \infty$ , all cumulants converge a same value,

$$\langle x(t)^n \rangle_c = N_{\text{eq}}, \quad (\text{B23})$$

which means that the distribution approaches the Poissonian in the large  $t$  limit irrespective of the initial condition.

## 3. Some examples of time evolution

It is instructive to see the time evolution of  $\langle x(t)^n \rangle_c$  with some specific initial conditions.

### a. Poisson distribution

First, let us consider the initial condition that  $P(x, t)$  is the Poisson distribution at  $t = 0$ . In this case, the initial condition satisfies

$$\langle N \rangle_0 = \langle \delta N^2 \rangle_0 = \langle \delta N^3 \rangle_0 = \langle \delta N^4 \rangle_{c,0}. \quad (\text{B24})$$

Substituting Eq. (B24) in Eq. (B19) - (B22), all terms in the right-hand side vanish except for the first two terms and one finds that the all cumulants have the same  $t$  dependence

$$\langle x(t)^n \rangle_c = N_{\text{eq}}(1 - e^{-k_1 t}) + \langle N \rangle_0 e^{-k_1 t}. \quad (\text{B25})$$

This result shows that the distribution  $P(x, t)$  starting from the Poissonian stays Poissonian for all  $t$  [39], while the average of the distribution shifts from  $\langle N \rangle_0$  at  $t = 0$  to  $N_{\text{eq}}$  for  $t \rightarrow \infty$ .

### b. $\langle N \rangle_0 = N_{\text{eq}}$

Next, we consider the initial condition

$$P(x, 0) = \delta_{x, N_{\text{eq}}}, \quad (\text{B26})$$

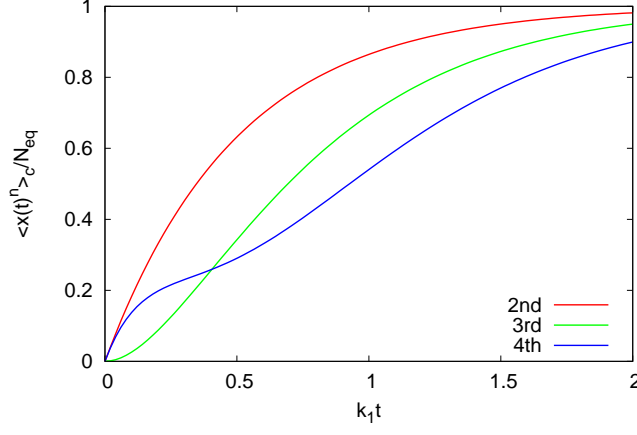


FIG. 9. Dependences of the cumulants  $\langle x(t)^n \rangle_c$  on  $t$  with fixed initial condition Eq. (B26).

i.e. the particle number is fixed to the equilibrated value  $N_{\text{eq}}$  without fluctuation. In this case, one obtains

$$\langle x(t) \rangle_c = N_{\text{eq}}, \quad (\text{B27})$$

$$\langle x(t)^2 \rangle_c = N_{\text{eq}}(1 - e^{-2k_1 t}), \quad (\text{B28})$$

$$\langle x(t)^3 \rangle_c = N_{\text{eq}}(1 - 3e^{-2k_1 t} + 2e^{-3k_1 t}), \quad (\text{B29})$$

$$\langle x(t)^4 \rangle_c = N_{\text{eq}}(1 - 7e^{-2k_1 t} + 12e^{-3k_1 t} - 6e^{-4k_1 t}). \quad (\text{B30})$$

In Fig. 9, we show the  $t$  dependence of second to fourth order cumulants for the initial condition Eq. (B26) up to fourth order. The figure shows that all cumulants approach  $N_{\text{eq}}$  as  $t$  increases. One also finds that the growth of higher order cumulants is slower. This behavior is understood as follows. First, the higher order cumulants are more sensitive to the tails, i.e. far side from the average value, of the distribution. Second, propagation of the probability distribution to far side from the  $N_{\text{eq}}$  should take longer time. Thus, the approach of higher order cumulants to the equilibrated value is much slower than the lower order ones.

### Appendix C: $I_X(\zeta)$ and $F_X$

In this appendix, we summarize property of  $I_X(\zeta)$  defined in Eq. (41)

$$I_X(\zeta) = \int_{-1/2}^{1/2} d\xi \int \frac{dp}{2\pi} e^{-X^2 p^2 / 2} e^{ip(\xi + \zeta)}, \quad (\text{C1})$$

and  $F_n(X)$  defined in Eq. (51).

By first integrating out  $p$  in Eq. (C1) one obtains

$$\begin{aligned} I_X(\zeta) &= \int_{-1/2}^{1/2} d\xi \frac{1}{\sqrt{2\pi X}} e^{-(\xi + \zeta)^2 / 2X^2} \\ &= \frac{1}{2} \left( \text{erf} \frac{\zeta + 1/2}{\sqrt{2}X} - \text{erf} \frac{\zeta - 1/2}{\sqrt{2}X} \right). \end{aligned} \quad (\text{C2})$$

For  $X = 0$ ,  $I_X(\zeta)$  satisfies

$$I_0(\zeta) = \int_{-1/2}^{1/2} d\xi \int \frac{dk}{2\pi} e^{ik(\xi + \zeta)} = \int_{-1/2}^{1/2} d\xi \delta(\xi - \zeta) = \theta(1 - \zeta^2), \quad (\text{C3})$$

where  $\theta(x)$  is the step function. From this result, one obtains

$$[I_0(\zeta)]^n = \theta(1 - \zeta^2), \quad (\text{C4})$$



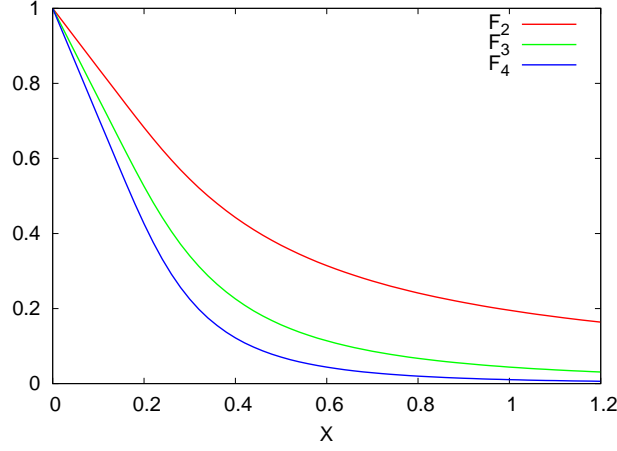


FIG. 10. Functions  $F_n(X)$  with  $n = 2, 3$  and  $4$ .

for arbitrary  $n$ .

In Sec. II, we define

$$F_n(X) = \int_{-\infty}^{\infty} dz [I_X(z/\Delta)]^n. \quad (\text{C5})$$

From Eq. (C4), one easily obtains

$$F_n(0) = 1 \quad (\text{C6})$$

for any  $n$ . The integral for  $n = 2$  can be performed analytically and one obtains

$$F_2(X) = \text{erf} \frac{1}{2X} - \frac{2X}{\sqrt{\pi}} (1 - e^{-1/4X^2}). \quad (\text{C7})$$

For  $n = 1$ ,  $F_1(X)$  is a constant

$$F_1(X) = 1. \quad (\text{C8})$$

For  $n \geq 2$ ,  $F_n(X)$  are monotonically decreasing functions of  $X$  with  $F_n(0) = 1$  and  $\lim_{X \rightarrow \infty} F_n(X) = 0$ . In Fig. 10, we show the  $X$  dependences of  $F_n(X)$  for  $n = 2, 3$  and  $4$ .

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